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EDINBURGH MATHEMATICAL SOCIETY.

THIRTEENTH SESSION, 1894–95.

First Meeting, November 9th, 1894.

Prof. KNOTT, D.Sc., F.R.S.E., President, in the Chair.

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Étude sur le Triangle et sur certains points de Géométrographie.

PAR M. ÉMILE LEMOINE.

Cette note que j'ai l'honneur de présenter à la Société Mathématique d'Edinburgh, par l'entremise aimable de M. J. S. Mackay, contient, ou des résultats que je crois nouveaux, ou des *développements* sur des sujets que j'ai déjà souvent abordés dans la Géométrie et qui concernent: *la transformation continue* dans le triangle et dans le tétraèdre, *les formules entre les éléments du triangle*, et la *Géométrographie*. Pour abréger, je passerai rapidement sur les points que j'ai déjà développés ailleurs, me contentant de renvoyer, si l'on désire plus d'explications, aux mémoires où la chose a été faite.

L'idée de la *transformation continue* n'est autre qu'une explicitation, pour le triangle et pour le tétraèdre, du principe de continuité de Carnot; elle permet de transformer les théorèmes, les formules, les équations qui déterminent les éléments de ces figures ou expriment leurs propriétés, de façon à ce que, sous le nouveau vêtement qu'elle leur fait prendre, on obtient des théorèmes, des formules, des équations nouvelles, mais qui ne sont, au fond, que des formes d'une même vérité.

Notations pour le triangle ABC et ses éléments.

Nous nommons a, b, c; A, B, C; $p, p-a, p-b, p-c$; S; R; r, r_a, r_b, r_c; $\delta, \delta_a, \delta_b, \delta_c$; ω; h_a, h_b, h_c; l_a, l_b, l_c; l_a', l_b', l_c' respectivement les trois côtés BC, CA, AB; les trois angles; les quantités $\frac{1}{2}(a+b+c)$, $\frac{1}{2}(b+c-a)$, $\frac{1}{2}(c+a-b)$, $\frac{1}{2}(a+b-c)$; la surface; le rayon du cercle circonscrit; les rayons du cercle inscrit et des trois cercles ex-inscrits; les quantités $4R+r$, $4R-r_a$, $4R-r_b$, $4R-r_c$; l'angle de Brocard; les trois hauteurs; les trois bissectrices intérieures; les trois bissectrices extérieures.

J'appelle x, y, z les coordonnées normales trilinéaires, par rapport au triangle de référence ABC.

J'appelle α, β, γ les coordonnées barycentriques par rapport à ce même triangle; et X, Y les coordonnées cartésiennes, *quels que soient les axes de coordonnées et leur angle.*

Voici un tableau qui donne le moyen d'opérer dans tous les cas, la transformation continue soit en A, soit en B, soit en C, dans les théorèmes, les formules, les équations.

Quantités.	Transformées en A.	Transformées en B.	Transformées en C.
a, b, c	$a, -b, -c$	$-a, b, -c$	$-a, -b, c$
A, B, C, ω	$-A, \pi-B, \pi-C, -\omega$	$\pi-A, -B, \pi-C, -\omega$	$\pi-A, \pi-B, -C, -\omega$
$p, (p-a), (p-b), (p-c)$	$-(p-a), -p, (p-c), (p-b)$	$-(p-b), (p-c), -p, (p-a)$	$-(p-c), (p-b), (p-c), -p$
S, R	$-S, -R$	$-S, -R$	$-S, -R$
r, r_a, r_b, r_c	$r_a, r, -r_c, -r_b$	$r_b, -r_c, r, -r_a$	$r_c, -r_b, -r_a, r$
$\delta, \delta_a, \delta_b, \delta_c$	$-\delta_a, -\delta, -\delta_c, -\delta_b$	$-\delta_b, -\delta_c, -\delta, -\delta_a$	$-\delta_c, -\delta_b, -\delta_a, -\delta$
h_a, h_b, h_c	$-h_a, h_b, h_c$	$h_a, -h_b, h_c$	$h_a, h_b, -h_c$
l_a, l_b, l_c	$-l_a, -l'_b, -l'_c$	$-l'_a, -l_b, -l'_c$	$-l'_a, -l'_b, -l_c$
l'_a, l'_b, l'_c	$l'_a, -l_b, -l_c$	$-l_a, l'_b, -l_c$	$-l_a, -l_b, l'_c$
x, y, z	$-x, y, z$	$x, -y, z$	$x, y, -z$
α, β, γ	α, β, γ	α, β, γ	α, β, γ
X, Y	$X, -Y$	$-X, Y$	$-X, -Y$

Si, dans une formule ou dans une équation se rapportant au triangle ABC et contenant les éléments marqués dans la première colonne de ce tableau indiquée par le mot: quantité, je remplace ces éléments par ceux qui leur correspondent dans une des trois colonnes suivantes, j'aurai opéré respectivement la *transformation continue* en A, en B, ou en C.

Dans une *transformation continue* :

I. La droite de l'infini se transforme en elle-même ; donc, deux droites parallèles restent parallèles dans la figure transformée.

II. Les points circulaires de l'infini se transforment l'un dans l'autre.

III. Le degré et la classe des courbes se conservent dans les courbes transformées.

IV. L'homographie, l'homologie, et l'orthologie ainsi que l'involution, se conservent.

V. Deux droites ou deux courbes qui sont orthogonales restent orthogonales.

VI. Si les longueurs de deux droites sont dans un rapport numérique indépendant des éléments du triangle, ce rapport se conservera.

La *transformation continue* faite en A, en B, et en C *successivement* présente les quatre cas suivants:

a. Elle ne change rien à la formule, à l'equation, au théorème que l'on considère, qui se reproduit.

Exemples: $a\cos B + b\cos A = c.$

Les trois hauteurs d'un triangle se coupent au même point dont les coordonnées sont

$$\frac{1}{\cos A},\ \frac{1}{\cos B},\ \frac{1}{\cos C}.$$

b. *A chacune des trois transformations correspond un résultat différent.*

Exemples : Au point qui a pour coordonnées

$$\frac{p-a}{a}, \ \frac{p-b}{b}, \ \frac{p-c}{c} \text{ (point de Nagel)}$$

correspondent respectivement les trois points :

$$-\frac{p}{a}, \ \frac{p-c}{b}, \ \frac{p-b}{c}; \quad \frac{p-c}{a}, \ -\frac{p}{b}, \ \frac{p-a}{c};$$

$\dfrac{p-b}{a}$, $\dfrac{p-a}{b}$, $-\dfrac{p}{b}$, avec des propriétés analogues à celles du point de Nagel.

L'axe anti-orthique (droite qui passe par les pieds des trois bissectrices extérieures et a pour équation $x+y+z=0$) devient par *transformation continue* en A, la droite qui passe par les pieds des bissectrices intérieures partant de B et de C et par le pied de la bissectrice extérieure partant de A ; son équation est $-x+y+z=0$. On appelle souvent cette droite l'interbissectrice relative à A. Résultats analogues par transformation en B et en C.

c. *Une des transformations conserve le résultat primitif, les deux autres le changent en un autre résultat, mais unique pour ces deux transformations.*

Exemple : La formule $S=\dfrac{ar_br_c}{\delta-r_a}$ se reproduit par transformation en A, mais transformée soit en B, soit en C, elle donne

$$S=\frac{arr_a}{4R-r_b-r_c}=\frac{arr_a}{r_a-r}$$

d. *La transformation continue faite soit en* A, *soit en* B, *soit en* C *donne un même résultat différent de celui qu'on transforme.*

Exemple : La conique inscrite qui a pour un de ses foyers le point d'où l'on voit les trois côtés sous le même angle (Conique de Simmons : voir J. J. Milne, *Companion to the Weekly Problem Papers*, p. 165), conique qui a pour équation

$$\sqrt{x\sin(A+60^\circ)}+\sqrt{y\sin(B+60^\circ)}+\sqrt{z\sin(C+60^\circ)}=0,$$

se transforme soit en A, soit en B, soit en C, en

$$\sqrt{x\sin(A-60^\circ)}+\sqrt{y\sin(B-60^\circ)}+\sqrt{z\sin(C-60^\circ)}=0.$$

Je n'ai pas trouvé de cas où l'une des transformations reproduise la formule et où les deux autres la modifient mais chacune d'une façon différente.

La transformation continue s'applique au tétraèdre. Nous allons donner seulement le tableau que permet la transformation des éléments de la figure.

Notations.

1°. *Je désigne par* A, B, C, D *les sommets du tétraèdre.*

2°. *Les faces* ABC, BCD, CDA, DAB *seront* F_d, F_a, F_b, F_c, *nous poserons* $F_a+F_b+F_c+F_d=S$.

3°. *Les angles plans des faces*

$$\left.\begin{array}{lll} BDC, & CDA, & ADB \\ CAB, & DAB, & DAC \\ ABC, & ABD, & DBC \\ BCA, & BCD, & DCA \end{array}\right\}\ \textit{seront}\ \left\{\begin{array}{lll} D_a, & D_b, & D_c \\ A_d, & A_c, & A_b \\ B_d, & B_c, & B_a \\ C_d, & C_a, & C_b \end{array}\right.$$

4°. *Les longueurs des côtés* BC, CA, AB, DA, DB, DC *seront* a, b, c, a', b', c'.

5°. *Les angles dièdres qui ont pour arêtes* BC, CA, AB, DA, DB, DC *seront* $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{a'}$, $\widehat{b'}$, $\widehat{c'}$.

Les angles que fait une arête avec les deux faces qui ne la contiennent pas seront désignés chacun par le lettre qui représente l'arête, suivie de la lettre qui représente la face considérée.

Il y aura donc les douze angles

$$\begin{array}{cccc} \widehat{aF_c}, & \widehat{aF_b}, & \widehat{a'F_d}, & \widehat{a'F_a} \\ \widehat{bF_c}, & \widehat{bF_a}, & \widehat{b'F_d}, & \widehat{b'F_b} \\ \widehat{cF_b}, & \widehat{cF_a}, & \widehat{c'F_d}, & \widehat{c'F_c} \end{array}$$

6°. *Les hauteurs seront* : h_a, h_b, h_c, h_d.

7°. *Les rayons et les centres de la sphère inscrite et des sphères ex-inscrites de* PREMIÈRE ESPÈCE *seront* :

$$r,\ r_a,\ r_b,\ r_c,\ r_d\ ;\quad o,\ o_a,\ o_b,\ o_c,\ o_d.$$

Les rayons et les centres des sphères ex-inscrites de SECONDE ESPÈCE, *ou sphères inscrites dans les combles du tétraèdre seront*

$$r_a',\ r_b',\ r_c'\ ;\quad o_a',\ o_b',\ o_c',$$

r_a' et o_a' ; r_b' et o_b' ; r_c' et o_c' appartenant *respectivement* à la sphère inscrite dans l'un des combles qui ont pour arêtes DA ou BC ; DB ou CA ; DC ou AB.

(On sait qu'il n'y a qu'une seule sphère pour deux combles opposés.)

Le tétraèdre *général* possède toujours ces huit sphères tangentes aux quatre faces ; quand une sphère ou deux sphères, ou les trois sphères inscrites dans les combles manquent, ce qui peut arriver, car lorsque la somme de deux faces qui ont même arête est égale à la somme des deux autres, la sphère des combles correspondant à ces arêtes a un rayon infini ; ce ne sont plus alors des tétraèdres *généraux* puisqu' il y a une ou plusieurs relations entre les faces, et la *transformation continue* n'est plus applicable, au moins sans discussion préalable.

8°. *Le volume du tétraèdre et le rayon de la sphère circonscrite seront* : V et R.

O sera le centre de cette sphère.

9°. *Les angles de* DA *avec* BC ; *de* DB *et de* AC ; *de* DC *et de* BA *seront* : α, β, γ.

10°. *Les longueurs des droites qui joignent les milieux de* DA *et de* BC ; *de* DB *et de* AC *de* DC *et de* BA *seront* : l, m, n.

De même que la *transformation continue* en A, dans le triangle, revient à changer a, b, c en $a, -b, -c$, la *transformation continue* en D, dans le tétraèdre, revient à changer a, b, c, a', b', c' en $a, b, c, -a', -b', -c'$.

Quantités.	Transformées en D.	Transformées en A.
a, b, c, a', b', c'	$a, b, c, -a', -b', -c'$	$a, -b, -c, -a', b', c'$
F_a, F_b, F_c, F_d	$-F_a, -F_b, -F_c, F_d$	$-F_a, -F_b, F_c, -F_d$
D_a, D_b, D_c	$-D_a, -D_b, -D_c$	$D_a, \pi - D_b, \pi - D_c$
A_d, A_c, A_b	$A_d, \pi - A_c, \pi - A_b$	$-A_d, A_c, -A_b$
B_d, B_c, B_a	$B_d, \pi - B_c, \pi - B_a$	$\pi - B_d, \pi - B_c, B_a$
C_d, C_a, C_b	$C_d, \pi - C_a, \pi - C_b$	$\pi - C_d, C_a, \pi - C_b$
$\widehat{a}, \widehat{b}, \widehat{c}, \widehat{a'}, \widehat{b'}, \widehat{c'}$	$\pi - \widehat{a}, \pi - \widehat{b}, \pi - \widehat{c}, \widehat{a'}, \widehat{b'}, \widehat{c'}$	$\pi - \widehat{a}, \widehat{b}, \widehat{c}, \widehat{a'}, \pi - \widehat{b'}, \pi - \widehat{c'}$
h_a, h_b, h_c, h_d	$h_a, h_b, h_c, -h_d$	h_a, h_b, h_c, h_d
$\widehat{a'F_d}, \widehat{a'F_a}, \widehat{aF_b}, \widehat{aF_c}$	$\pi - \widehat{a'F_d}, -\widehat{a'F_a}, \widehat{aF_b}, \widehat{aF_c}$	$\widehat{a'F_d}, \pi - \widehat{a'F_a}, \widehat{aF_b}, \widehat{aF_c}$
$\widehat{b'F_d}, \widehat{bF_a}, \widehat{b'F_b}, \widehat{bF_c}$	$\pi - \widehat{b'F_d}, \widehat{bF_a}, -\widehat{b'F_b}, \widehat{bF_c}$	$\widehat{b'F_d}, \pi - \widehat{bF_a}, \widehat{b'F_b}, -\widehat{bF_c}$
$\widehat{c'F_d}, \widehat{cF_a}, \widehat{cF_b}, \widehat{c'F_c}$	$\pi - \widehat{c'F_d}, \widehat{cF_a}, \widehat{cF_b}, -\widehat{c'F_c}$	$\widehat{c'F_d}, \pi - \widehat{cF_a}, -\widehat{cF_b}, \widehat{c'F_c}$
r, r_a, r_b, r_c, r_d	r_d, r_a', r_b', r_c', r	$r_a, r, -r_c', -r_b', r_a'$
r_a', r_b', r_c'	r_a, r_b, r_c	$r_d, -r_c, -r_b$

Quantités.	Transformées en B.	Transformées en C.
$a,\ b,\ c,\ a',\ b',\ c'$	$-a,\ b,\ -c,\ a',\ -b',\ c'$	$-a,\ -b,\ c,\ a',\ b',\ -c'$
$F_a,\ F_b,\ F_c,\ F_d$	$-F_a,\ F_b,\ -F_c,\ -F_d$	$-F_a,\ -F_b,\ F_c,\ -F_d$
$D_a,\ D_b,\ D_c$	$\pi-D_a,\ D_b,\ \pi-D_c$	$\pi-D_a,\ \pi-D_b,\ D_c$
$A_d,\ A_c,\ A_b$	$\pi-A_d,\ \pi-A_c,\ A_b$	$\pi-A_d,\ A_c,\ \pi-A_b$
$B_d,\ B_c,\ B_a$	$-B_d,\ -B_c,\ -B_a$	$\pi-B_d,\ B_c,\ \pi-B_a$
$C_d,\ C_a,\ C_b$	$\pi-C_d,\ \pi-C_a,\ C_b$	$-C_d,\ -C_a,\ -C_b$
$\widehat{a},\ \widehat{b},\ \widehat{c},\ \widehat{a'},\ \widehat{b'},\ \widehat{c'}$	$\widehat{a},\ \pi-\widehat{b},\ \widehat{c},\ \pi-\widehat{a'},\ \widehat{b'},\ \pi-\widehat{c'}$	$\widehat{a},\ \widehat{b},\ \pi-\widehat{c},\ \pi-\widehat{a'},\ \pi-\widehat{b'},\ \widehat{c'}$
$h_a,\ h_b,\ h_c,\ h_d$	$h_a,\ -h_b,\ h_c,\ h_d$	$h_a,\ h_b,\ h_c,\ -h_d$
$\widehat{a'F_d},\ \widehat{a'F_a},\ \widehat{aF_b},\ \widehat{aF_c}$	$\widehat{a'F_d},\ \widehat{a'F_a},\ \pi-\widehat{aF_b},\ -\widehat{aF_c}$	$\widehat{a'F_d},\ \widehat{a'F_a},\ -\widehat{aF_b},\ \pi-\widehat{aF_c}$
$\widehat{b'F_d},\ \widehat{bF_a},\ \widehat{b'F_b},\ \widehat{bF_c}$	$-\widehat{b'F_d},\ \widehat{bF_a},\ \pi-\widehat{b'F_b},\ \widehat{bF_c}$	$\widehat{b'F_d},\ -\widehat{bF_a},\ \widehat{b'F_b},\ \pi-\widehat{bF_c}$
$\widehat{c'F_d},\ \widehat{cF_a},\ \widehat{cF_b},\ \widehat{c'F_c}$	$\widehat{c'F_d},\ -\widehat{cF_a},\ \pi-\widehat{cF_b},\ \widehat{c'F_c}$	$-\widehat{c'F_d},\ \widehat{cF_a},\ \widehat{cF_b},\ \pi-\widehat{c'F_c}$
$r,\ r_a,\ r_b,\ r_c,\ r_d$	$r_b,\ r_c',\ r,\ r_a',\ r_b'$	$r_c,\ -r_b',\ -r_a',\ r,\ r_c'$
$r_a',\ r_b',\ r_c'$	$-r_c,\ r_d,\ -r_a$	$-r_b,\ -r_a,\ r_d$

V et R se changent en : $-V$ et $-R$ dans les quatre transformations.

α, β, γ deviennent : $\pi-\alpha,\ \pi-\beta,\ \pi-\gamma$ dans les quatre transformations.

l, m, n ne changent pas.

Nous avons vu par le tableau relatif au triangle qu'à un point donné $M(x, y, z)$ dans le plan du triangle peuvent correspondre trois *transformés continus* M_a, M_b, M_c dont les coordonnées sont

$$-x_a, y_a, z_a\,;\ x_b, -y_b, z_b\,;\ x_c, y_c, -z_c$$

en affectant des indices a, b, c les quantités qui représentent ce que deviennent les coordonnées quand on les transforme en A, en B, et en C. Par rapport au tétraèdre un point $M(x, y, z, t)$ peut avoir sept transformés continus dont les coordonnées sont

1°. Quatre transformés de première espèce

$$x_d, y_d, z_d, -t_d\,;\ -x_a, y_a, z_a, t_a\,;\ x_b, -y_b, z_b, t_b\,;\ x_c, y_c, -z_c, t_c$$

2°. Trois transformés de seconde espèce

$$-x_{da}, y_{da}, z_{da}, -t_{da}\,;\ x_{db}, -y_{db}, z_{db}, -t_{db}\,;\ x_{dc}, y_{dc}, -z_{dc}, -t_{dc}$$

x_d désignant ce que devient x par *transformation continue* en D, x_{da} désignant ce que devient x si l'on fait d'abord la *transformation continue* en D sur lui, ce qui donne x_d; puis la *transformation continue* en A sur x_d, ce qui donne x_{da}, etc.

Pour les démonstrations et l'exposition de la *transformation continue* nous renvoyons aux mémoires suivants : *Association Française pour l'avancement des Sciences, Congrès de Marseille* 1891 ; *Mathesis*, 1892, pp. 58–64 81–92 ; *Nouvelles Annales de Mathématiques*, *Janvier* 1893, etc ; et pour l'application au tétraèdre, *Association Française, Congrès de Besançon*, 1893.

Nous allons donner maintenant quelques exemples des applications de la *transformation continue.* Cela nous fournira en même temps l'occasion d'attirer l'attention sur l'emploi de formules symétriques entre les éléments du triangle, emploi qui est très avantageux pour effectuer beaucoup de calculs qui, à première vue, paraîtraient inextricables.

Je me propose, d'abord, de calculer les huit rayons des cercles qui sont tangents aux trois cercles décrits des sommets A, B, C d'un triangle comme centres avec $BC = a$, $CA = b$, $AB = c$ comme rayons.

Prenons la question générale de la recherche des rayons des cercles tangents à trois cercles donnés de centres A, B, C et de rayons que j'appelle R_a, R_b, R_c. Ce que nous allons faire ainsi est

une manière de traiter le célèbre problème d'Apollonius, manière qui, je crois, n'a pas encore été considérée. Pour fixer les idées nous supposons, sur la figure (Fig. 1) qu'il s'agit de chercher le rayon ρ du cercle de centre o qui a les trois cercles donnés à l'extérieur.

Joignons oA, oB, oC que nous appelons X, Y, Z ; pour tout point du plan on a la relation

$$\Sigma a^2 X^4 \quad \Sigma(b^2+c^2-a^2)(a^2X^2+Y^2Z^2)+a^2b^2c^2=0$$

entre les distances X, Y, Z d'un point quelconque aux trois sommets.

D'ailleurs, comme on a :

$$X=R_a+\rho,\ Y=R_b+\rho,\ Z=R_c+\rho$$

on peut écrire

$$\Sigma a^2(\rho+R_a)^4-\Sigma(b^2+c^2+c^2)\{a^2(\rho+R_a)^2+(\rho+R_b)^2(\rho+R_c)^2\}+a^2b^2c^2=0$$

Si l'on développe cette équation en l'ordonnant par rapport à ρ, on voit très facilement que les coëfficients des termes en ρ^4 et en ρ^3 sont nuls identiquement et il vient

$$\begin{aligned}(1)\qquad &\rho^2\Sigma[6a^2R_a^{\,2}-(b^2+c^2-a^2)\{a^2+(R_b+R_c)^2+2R_bR_c\}]\\ &+2\rho\,\Sigma[2a^2R_a^{\,3}-(b^2+c^2-a^2)\{a^2R_a+R_b\,R_c\,(R_b+R_c)\}]\\ &+a^2b^2c^2+\Sigma\,\{a^2R_a^{\,4}-(b^2+c^2-a^4)\,(a^2R_a^{\,2}+R_b^{\,2}R_c^{\,2})\}=0\end{aligned}$$

Cette équation donnera deux valeurs ρ' et ρ'' correspondant aux cercles qui touchent les trois cercles donnés en les ayant tous les trois à l'extérieur, ou tous les trois à l'intérieur.

Pour revenir au problème que je me proposais de résoudre, il faut faire : $\qquad R_a=a,\ R_b=b,\ R_c=c$

et calculer le coëfficient de ρ^2, celui de ρ, et le terme indépendant de ρ. Appelons L, M, N ces coëfficients, il faut évaluer maintenant

$$\begin{aligned}L&=\Sigma[6a^4-(b^2+c^2-a^2)\{a^2+(b+c)^2+2bc\}]\\ M&=\Sigma[2a^5-(b^2+c^2-a^2)\{a^3+bc(b+c)\}]\\ N&=a^2b^2c^2+\Sigma\{a^6-(b^2+c^2-a^2)(a^4+b^2c^2)\}.\end{aligned}$$

Calcul de L. On peut écrire

$$L = 6\Sigma a^4 - \Sigma(a^2+b^2+c^2-2a^2)(a^2+b^2+c^2+4bc), \text{ ou}$$

$$L = 6\Sigma a^4 - \Sigma(a^2+b^2+c^2)^2 + 2\Sigma a^2(a^2+b^2+c^2) - 4\Sigma bc(a^2+b^2+c^2) + 8\Sigma a^2bc, \text{ ou}$$

$$L = 6\Sigma a^4 - 3(a^2+b^2+c^2)^2 + 2(a^2+b^2+c^2)\Sigma a^2 - 4(a^2+b^2+c^2)\Sigma bc + 8abc\Sigma a, \text{ ou}$$

$$L = 6\Sigma a^4 - 2(a^2+b^2+c^2)^2 - 4(a^2+b^2+c^2)\Sigma bc + 16pabc.$$

Mais les formules dont je parlais tout à l'heure (voir *Mathesis* 1892, loco citato) donnent

$$\Sigma a^4 = 2\{p^2 - r\delta)^2 - 4S^2\}, \quad a^2+b^2+c^2 = 2(p^2-r\delta)$$

$$\Sigma bc = p^2 + r\delta\,;$$

on a d'ailleurs, $abc = 4RS$ et $S = pr$.

Substituant et effectuant les calculs, on trouve très facilement

$$L = 16r^2(\delta^2 - 4p^2)$$

Calcul de M. On a :

$$M = 2\Sigma a^5 - \Sigma(b^2+c^2-a^2)a^3 - \Sigma bc(b+c)(b^2+c^2-a^2), \text{ ou}$$

$$M = 2\Sigma a^5 - \Sigma(a^2+b^2+c^2-2a^2)a^3 - \Sigma bc(b+c)(a^2+b^2+c^2-2a^2), \text{ ou}$$

$$M = 4\Sigma a^5 - (a^2+b^2+c^2)\Sigma a^3 - (a^2+b^2+c^2)\Sigma bc(b+c) - 2abc\Sigma(b+c)a, \text{ ou}$$

$$M = 4\Sigma a^5 - (a^2+b^2+c^2)\Sigma a^3 - (a^2+b^2+c^2)\Sigma bc(a+b+c-a)4abc\Sigma bc, \text{ ou}$$

$$M = 4\Sigma a^5 - (a^2+b^2+c^2)\Sigma a^3 - 2p(a^2+b^2+c^2)\Sigma bc + 3abc(a^2+b^2+c^2) - 4abc\Sigma bc$$

Les formules déjà citées donnent

$$\Sigma a^3 = 2p(p^2 + 6Rr - 3r\delta) \quad = 2p\{p^2 - 3r(2R+r)\}$$

$$\Sigma a^5 = 2p\{p^4 - 10p^2r(R+r) + 5r^2\delta(2R+r)\}$$

Substituant et réduisant, on trouve

$$M = 64pr^2\{\delta(2R+r) - 2p^2\}$$

Calcul de N. On a

$$N = a^2b^2c^2 + \Sigma\{a^6 - (b^2 + c^2 - a^2)(a^4 + b^2c^2)\} \quad \text{ou}$$

$$N = a^2b^2c^2 + \Sigma a^6 - \Sigma(a^2 + b^2 + c^2 - 2a^2)(a^4 + b^2c^2) \quad \text{ou}$$

$$N = a^2b^2c^2 + \Sigma a^6 - (a^2 + b^2 + c^2)\Sigma a^4 + 2\Sigma a^6 - (a^2 + b^2 + c^2)\Sigma b^2c^2 + 2\Sigma a^2b^2c^2 \quad \text{ou}$$

$$N = 7a^2b^2c^2 + 3\Sigma a^6 - (a^2 + b^2 + c^2)\Sigma a^4 - (a^2 + b^2 + c^2)\Sigma b^2c^2$$

On trouve dans nos formules

$$\Sigma b^2c^2 = (p^2 - r\delta)^2 + 4S^2$$

On n'y trouve pas Σa^6 mais il peut se calculer aisément en partant de Σa^4 et de Σa^2.

On a en effet

$$\begin{aligned}(a^4 + b^4 + c^4)(a^2 + b^2 + c^2) &= \Sigma a^6 + \Sigma a^2(b^4 + c^4)\\ &= \Sigma a^6 + \Sigma b^2c^2(b^2 + c^2) \quad \text{ou}\end{aligned}$$

$$\begin{aligned}4\{(p^2 - r\delta)^2 - 4S^2\}(p^2 - r\delta) &= \Sigma a^6 + \Sigma b^2c^2(a^2 + b^2 + c^2 - a^2)\\ &= \Sigma a^6 + 2(p^2 - r\delta)\Sigma b^2c^2 - 3a^2b^2c^2 \quad \text{ou}\end{aligned}$$

$$\begin{aligned}4(p^2 - r\delta)^3 - 16p^2r^2(p^2 - r\delta) &= \Sigma a^6 + 2(p^2 - r\delta)\{(p^2 - r\delta)^2 + 4S^2\}\\ &\quad - 48p^2R^2r^2\end{aligned}$$

d'où $$\Sigma a^6 = 2(p^2 - r\delta)^3 - 24p^2r^2(p^2 - r\delta) + 48p^2R^2r^2$$

Substituant on trouve

$$N = 64p^2r^2\{(2R + r)^2 - p^2\}$$

Remarquons, en passant, que l'on a

$$p^2 - (2R + r)^2 = 4R^2\cos A\cos B\cos C.$$

L'équation qui détermine ρ devient alors

$$^2(\delta^2 - 4p^2) + 4p\rho\{\delta(2R + r) - 2p^2\} - 4p^2\{p^2 - (2R + r)^2\} = 0$$

On en tire, toutes réductions faites :

$$\rho' = \frac{2p(2R+r-p)}{2p-\delta} \quad \rho'' = -\frac{2p(2R+r+p)}{2p+\delta} \tag{2}$$

La formule (1) lorsque les racines sont réelles donne toujours les rayons avec un signe, naturellement ; mais il faut interpréter géométriquement ce signe. Ainsi, avec les deux formules précédentes, si le triangle est équilatéral, on trouve

$$\rho' = -\frac{\sqrt{3}-1}{\sqrt{3}}, \quad \rho'' = -\frac{\sqrt{3}+1}{\sqrt{3}}$$

et ce sont bien, mais en *valeur absolue* seulement, les valeurs des rayons qui conviennent pour ce cas, comme la géométrie le montre immédiatement. (Voir la note additionnelle à la fin du mémoire.)

Si l'on applique la *Transformation continue* en A aux formules (2), il vient

$$\rho_a' = \frac{2(p-a)\{-2R+r_a+(p-a)\}}{2(p-a)-\delta_a},$$

$$\rho_a'' = -\frac{2(p-a)\{-2R+r_a+(p-a)\}}{2(p-a)+\delta_a}$$

qui donnent les rayons du couple de cercles tangents aux trois cercles donnés, le premier tangent à l'intérieur du cercle de centre A et à l'extérieur des deux autres, le second tangent à l'extérieur du cercle de centre A et à l'interieur des deux autres ; on aurait de même les rayons des cercles des deux autres couples par *transformation continue* en B et en C.

Ces calculs paraissent fort longs surtout parceque nous les avons développés, dans le but de montrer, pour des cas analogues, notre manière d'opérer, mais ils sont très symétriques, très aisés, et nous ne savons d'ailleurs pas comment on aurait pu arriver à ces résultats sans nos formules et sans la *Transformation continue* ; il ne serait probablement même point facile d'y arriver synthétiquement en supposant ces résultats connus. et en cherchant alors à les démontrer.

Remarques.

L'axe de similitude externe des trois circonférences

$$A(a),\ B(b),\ C(c) \ \text{ est la droite } \ a^2x + b^2y + c^2z = 0.$$

On en conclut, par *transformation continue* en A, que la droite (coordonnées normales) $-a^2x + b^2y + c^2z = 0$ est l'axe de similitude qui passe par les centres de similitude interne de $A(a)$ et $B(b)$, et de $A(a)$ et $C(c)$. Nous désignons par $M(R)$ un cercle de rayon R et de centre M.

Le centre radical de ces trois circonférences est le point dont les coordonnées normales sont :

$$\cos A - \cos B\cos C,\ \ \cos B - \cos C\cos A,\ \ \cos C - \cos A\cos B$$

c'est un point que l'on rencontre assez souvent dans la Géométrie du triangle et qui est le symétrique de l'orthocentre par rapport au centre du cercle circonscrit.

On trouve très simplement les coordonnées de ce centre radical ; en effet (Association française, 1888, Congrès d'Oran, p. 170 VI.) il a pour coordonnées normales

$$abc\cos A - a^3 + b^3\cos C + c^3\cos B, \ \text{etc.}$$

Mais $$b^3\cos C + c^3\cos B - a^3 = 4RS(\cos A - 2\cos B\cos C)\ ;$$

elles deviennent donc

$$4RS\cos A + 4RS(\cos A - 2\cos B\cos C), \ \text{etc.}$$

ou $\cos A - \cos B\cos C$, comme nous l'avons dit.

La remarque permet de placer très simplement ce point dans le triangle ABC.
Pour cela je trace les trois cercles $A(a)$, $B(b)$, $C(c)$ op : $(9C_1 + 3C_3)$
et comme ces cercles se rencontrent deux à deux, il suffit de tracer deux de leurs intersections op : $(4R_1 + 2R_2)$
qui se coupent au point cherché.
En tout op : $(4R_1 + 2R_2 + 9C_1 + 3C_3)$

Simplicité 18 ; exactitude 13 ; 2 droites, 3 cercles.

Sans entrer dans d'autres détails j'énoncerai encore les applications suivantes de la transformation continue et de nos formules.

Les points ω, ω' qui ont pour coordonnées normales respectivement

$$\frac{a+r_a}{a}, \quad \frac{b+r_b}{b}, \quad \frac{c+r_c}{c}; \qquad \frac{a-r_a}{a}, \quad \frac{b-r_b}{b}, \quad \frac{c-r_c}{c}$$

sont des points qui jouissent de propriétés remarquables et que j'ai souvent rencontrés, ainsi que leurs transformés continus en A: ω_a, ω_a'

$$\frac{a+r}{a}, \quad \frac{b+r_c}{b}, \quad \frac{c+r_b}{c}; \qquad \frac{a-r}{a}, \quad \frac{b-r_c}{b}, \quad \frac{c-r_b}{c};$$

en B : ω_b, ω_b', etc.

Ces huit points $\omega, \omega', \omega_a, \omega_a'$, etc., sont les centres des quatre couples de cercles tangents savoir :

ω, ω' aux cercles $A(p-a), B(p-b), C(p-c)$ tangents deux à deux ;

ω_a, ω_a' aux cercles $A(p), B(p-c), C(p-a)$;

ω_b, ω_b' etc.

On peut trouver les rayons de ces cercles en appliquant la formule générale que nous avons donnée plus haut pour résoudre le problème d'Apollonius. Il suffit de faire :

pour ω et ω', $R_a = p-a$, $R_b = p-b$, $R_c = p-c$

pour ω_a et ω_a', $R_a = p$, $R_b = p-c$, $R_c = p-b$

et de réduire les coefficients de l'équation en ρ^2 par une méthode analogue à celle que nous avons donnée. On trouvera pour les cercles de centres ω et ω'

$$\rho = \frac{S}{2p+\delta}, \quad \rho' = \frac{S}{2p-\delta}, \text{ etc.}$$

On a : $$\overline{\omega\,\omega'}^2 = 16\mathrm{S}^2 \frac{\delta^2 - 3p^2}{(4p^2 - \delta^2)^2} \;;$$

et par transformation continue en A

$$\overline{\omega_a\omega_a'}^2 = 16\mathrm{S}^2 \frac{\delta_a{}^2 - 3(p-a)^2}{\{4(p-a)^2 - \delta_a{}^2\}^2}$$

Les coordonnées cartésiennes du centre O du cercle inscrit par rapport à HB pris pour axe des x et à HA pour axe des y sont, comme il est facile de le voir, (H étant l'orthocentre)

$$x = \frac{c\cos\mathrm{B} - (p-b)}{\sin\mathrm{C}} \qquad y = \frac{r\sin\mathrm{C} - (p-b)\cos\mathrm{C}}{\sin\mathrm{C}}$$

En appliquant la *transformation continue* en A, en B, en C à ces expressions de coordonnées on a immédiatement les coordonnées, par rapport à ces mêmes axes, des centres O_a, O_b, O_c des cercles exinscrits.

O_a : $$x = \frac{c\cos\mathrm{B} - (p-c)}{\sin\mathrm{C}}, \quad y = -\frac{r_a\sin\mathrm{C} + (p-c)\cos\mathrm{C}}{\sin\mathrm{C}}$$

O_b : $$x = \frac{c\cos\mathrm{B} - p}{\sin\mathrm{C}}, \quad y = \frac{r_b\sin\mathrm{C} - p\cos\mathrm{C}}{\sin\mathrm{C}}$$

O_c : $$x = \frac{c\cos\mathrm{B} + (p-a)}{\sin\mathrm{C}}, \quad y = \frac{r_c\sin\mathrm{C} + (p-a)\cos\mathrm{C}}{\sin\mathrm{C}}$$

L'équation, en coordonnées normales, de l'ellipse qui a pour foyers deux sommets du triangle, B et C par exemple, et passe par le troisième en A, est

$$p(p-a)(b^2y^2 + c^2z^2) + bcyz\{p^2 + (p-a)^2\} + abcx(b+c)(y+z) = 0$$

Si on la transforme en A, elle se reproduit, mais si on la transforme soit en B, soit en C, on obtient l'équation de l'hyperbole qui a pour foyers B et C et passe en A

$$(p-b)(p-c)(b^2y^2 + c^2z^2) - bcyz\{(p-b)^2 + (p-c)^2\} + abcx(b-c)(y-z) = 0$$

Je vais donner quelques explications sur la façon dont j'ai obtenu les nombreuses formules auxquelles je fais souvent allusion ici, et que j'ai employées, sans les démontrer ; elles dérivent des formules connues

$$S = pr, \text{ etc.}, \quad r_a + r_b + r_c = 4R + r,$$
$$p(p-a) = r_b r_c, \quad (p-b)(p-c) = r r_a, \text{ etc.}$$

et de quelques autres que j'ai rencontrées, et qui ne l'étaient pas ou du moins dont on n'avait pas remarqué la fécondité. Je citerai par exemple les trois suivantes

$$\cos A = \frac{2R + r - r_a}{2R}$$
$$a^2 + b^2 + c^2 = 2(p^2 - r\delta)$$
$$bc + ca + ab = p^2 + r\delta$$

La première peut se démontrer ainsi.

Soient x, y, z les perpendiculaires abaissées du centre du cercle circonscrit sur les trois côtés, on a :

$$(1) \qquad x + y + z = R + r$$

C'est un théorème de Carnot dont M. J. S. Mackay a donné de nombreuses démonstrations dans son intéressant mémoire *The Triangle and its Six Scribed Circles* (Edinburgh Mathematical Society, 1883).

Si nous transformons l'équation (1) continûment en A elle devient

$$(2) \qquad -x + y + z = -R + r_a.$$

De (1) et de (2) on déduit

$$x = \frac{2R + r - r_a}{2}$$

Mais $\qquad x = R\cos A \qquad$; donc

$$\cos A = \frac{2R + r - r_a}{2R}.$$

Je ne vois pas de moyen de démontrer les deux autres formules par des considérations géométriques simples sur une figure,

et *ce serait à désirer*, mais voici la façon de les obtenir ensemble. On a

$$r_a + r_b + r_c \;=\; 4R + r \;=\; \delta$$

d'où
$$\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} = \frac{2S}{\delta}$$

puisque $S = r_a(p-a)$, etc.

De là je tire

$$\frac{a^2-(b-c)^2+b^2-(c-a)^2+c^2-(a-b)^2}{16S^2} = \frac{4Sp}{\delta}$$

d'où
$$2\Sigma bc - \Sigma a^2 \qquad\qquad = 4\delta r$$

Mais on a identiquement

$$2\Sigma bc + \Sigma a \qquad\qquad = 4p^2$$

d'où l'on tire

$$\Sigma bc = p^2 + r\delta \quad \text{et} \quad \Sigma a^2 = 2(p^2 - r\delta)$$

Je bornerai là ce que je veux dire de ces formules, mais puisque c'est M. J. S. Mackay qui me fait l'honneur de présenter cette note à la Société Mathématique d'Edinburgh, je veux aussi ajouter quelques observations relatives à la Géométrographie qu'il vous a fait connaître, il y a quelques mois, en l'appliquant devant vous à la recherche du symbole, de la simplicité et de l'exactitude des constructions données dans l'Euclide, employé presque universellement, en Angleterre, pour l'étude des éléments de Géométrie.

Nous sommes sur le sujet, tout à fait d'accord, nous ne différons que sur des détails très peu importants au fond; seulement en dehors de l'avantage sérieux d'avoir partout identiquement les mêmes notations, je pense que celles que j'emploie, sont plus dans l'esprit de la méthode telle que je l'ai conçue, et je vais essayer brièvement de le convaincre.

Je rappelle que l'essence de la Géométrographie est spéculative, elle ne s'applique aux constructions à effectuer que—*si parva licet componere magnis*—comme la Mécanique rationnelle s'applique à l'art de l'Ingénieur. Voici mes notations

1. Faire passer le bord d'une règle par un point placé, c'est *l'opération* (R_1)

donc, *spéculativement*, faire passer le bord d'une règle par deux points placés, c'est op : ($2R_1$)

2. Tracer la droite qui suit le bord d'une règle, c'est op : (R_2)

3. Mettre une pointe d'un compas en un point placé, c'est op : (C_1)

donc, *spéculativement*, prendre avec le compas une longueur placée, c'est op : ($2C_1$)

4. Mettre une pointe en un point indéterminé d'une ligne tracée, c'est op : (C_2)

5. Tracer le cercle, c'est op : (C_3)

Monsieur Mackay supprime l'opération C_2 qu'il assimile à C_1, parceque, dit-il, quand on met la pointe en un point indéterminé d'une ligne on vise d'abord le point où l'on veut placer la pointe, c'est à dire qu'on la met réellement en un point déterminé; enfin l'on a ainsi l'avantage d'avoir plus de symétrie dans les symboles, puisque, supprimant mon symbole C_2, il appelle C_2 ce que j'appelle C_3 et que l'on a : R_1, R_2 symboles pour la droite, C_1, C_2 symboles pour le cercle.

Monsieur Mackay a raison, l'on vise effectivement, en pratique, un point avant d'y poser la pointe, mais peu importe; au point de vue spéculatif de la Géométrographie, mettre une pointe en un point déterminé et mettre une pointe en un point indéterminé d'une ligne, sont deux choses *essentiellement* différentes; c'est pour cela que je crois préférable de conserver les notations C_1 et C_2 afin de marquer la différence, *puisque*, d'ailleurs, il n'y a aucun inconvénient à la chose. Quant à l'avantage de la symétrie, il serait grand, si l'on avait à faire des calculs avec les symboles comme avec les expressions algébriques, mais comme le symbole final est simplement un tableau qui ne sert qu'à lire des résultats, il suffit qu'il soit clair et la suppression du symbole C_2 n'y apporte, pas plus de clarté et fait disparaître un détail des phases de l'opération. La symétrie pour l'oeil, à un point de vue en quelque sorte esthétique, n'est dans ce cas qu'une illusion, car elle ne s'appliquerait qu'à une partie de la

Géométrographie, celle qui étudie les constructions canoniques de la règle et du compas, elle disparaîtrait avec l'adjonction des symboles nouveaux qu'amène l'emploi de l'équerre, ainsi que je l'ai développé dans un mémoire présenté au mois d'Août cette année au Congrès de l'Association Française, à Caen. Ajoutons que la question reste, en réalité, presque du domaine de la théorie, car le symbole C_2 se rencontre assez rarement dans les constructions et généralement par faibles unités dans celles où il se rencontre.

Enfin nous avons apprécié différemment dans un cas—très peu important également—la manière de compter les symboles.

Pour tracer deux droites AB, AC *passant par un même point*, je ne tiens pas compte de ce qu'elles passent par le même point et j'évalue le tracé $4R_1 + 2R_2$ comme s'il s'agissait de deux droites différentes AB, CD. Cela, par la raison qu'on ne peut maintenir la règle en A (une fois qu'on a tracé AB). Pour tracer AC on recommence simplement l'opération sans que la nouvelle opération profite en quoique ce soit de ce qu'on a fait pour la première. Monsieur Mackay compte $2R_1 + R_2$ pour la première et seulement $R_1 + R_2$ pour la seconde, comme si la règle avait tourné autour de A. Je m'aperçois très bien que je m'appuie sur deux ordres de raisons qui semblent se contredire; en effet pour justifier l'emploi de C_2, je m'appuie sur l'essence spéculative de la Géométrographie et, pour justifier mon évaluation du tracé successif de deux droites qui ont un point commun, j'invoque la manœuvre pratique que l'on exécute. Celà est fort naturel cependant, parceque, quoique spéculative, la Géométrographie a en vue l'application possible de ses spéculations, elle a donc deux faces et, si elle doit rester tout à fait spéculative *lorsque cela n'a aucun inconvénient*,—c'est ainsi que pour tracer le cercle $O(b)$ immédiatement après avoir tracé le cercle $O(a)$, je ne compte que C_3, si je n'ai pas eu à déplacer la pointe fixée en O, mais seulement à ouvrir ou à fermer la branche du compas, parceque *je puis* opérer ainsi—il me semble utile de condescendre à la pratique dans le cas où j'ai à tracer deux droites successives ayant un point commun *puisque le contraire aurait un inconvénient* et que je suis, de plus près, la manœuvre du tracé.

Je ne m'étendrai pas sur l'emploi que j'ai fait de l'équerre en appliquant la Géométrographie à la Géométrie descriptive, je renvoie pour cela à mon mémoire déjà cité, présenté au Congrès de Caen. Je vais seulement définir les nouveaux symboles que j'ai adoptés en les ajoutant aux anciens.

Je suppose que l'on emploie l'équerre seulement pour tracer les parallèles et jamais pour tracer les perpendiculaires ; je ne suis point habitué à la pratique du dessin, mais je sais que cette manière d'opérer est proscrite dans toutes les épures exactes. Par exemple les instructions données aux dessinateurs des services de la Ville de Paris recommandent de ne jamais tracer de perpendiculaires avec l'équerre ; si l'on a plusieurs perpendiculaires à tracer à une même direction, on détermine la première avec la règle et le compas et les autres avec l'équerre, comme étant parallèles à celle-ci.

Nous admettons cependant que l'on se serve d'un té dont la direction donne les parallèles à la ligne de terre sur la feuille collée sur la planche, et alors, que l'on puisse mener, à l'équerre, les lignes de rappel relatives à cette ligne de terre, après avoir vérifié l'exactitude du tracé une fois pour toutes.

On n'a donc, pour l'usage de l'équerre, que les opérations nouvelles suivantes (sans compter celles où l'équerre sert comme servirait la règle seule, opérations qui conservent le symbole adopté).

—*Mettre un bord de la règle ou de l'équerre en coincidence avec une droite déjà tracé sur la figure.* Nous pourrions assimiler cette opération à faire passer le bord par deux points et compter alors $2R_1$, mais il nous semble utile—et sans aucun inconvénient—de distinguer, légèrement, cette nouvelle opération, afin de pouvoir apprécier, d'un coup d'oeil jeté sur le symbole final de la construction, le degré de fréquence dans l'emploi qu'on y a fait de l'équerre ; aussi nous l'appellerons op : $(2R_1')$

Nous ne comptons pas l'acte tout mécanique de placer l'équerre contre la règle ou réciproquement.

—Faire glisser l'équerre jusqu' à ce que son bord passe en un point placé op : (E)

R_1' et E sont les seuls symboles nouveaux à employer et, toute opération faite avec le compas, la règle et l'équerre, aura pour symbole :

$$\text{Op} : (l_1R_1 + l_2R_1' + l_3R_2 + l_4E + l_5C_1 + l_6C_2 + l_7C_3)$$

$l_1 + l_2 + \ldots + l_7$ sera le coëfficient de simplicité

$l_1 + l_2 + l_4 + l_5 + l_6$ le coëfficient d'exactitude

l_3 sera le nombre de droites tracées

l_7 celui des cercles.

Je veux terminer en vous donnant les résultats d'une expérience que la complaisance de M. Jung, le savant professeur de Milan, m'a permis d'exécuter et qui montre jusqu' où peut aller la simplification que la Géométrographie introduit dans une construction, simplification qui est due surtout à l'idée fondamentale de la Géométrographie, c'est à dire à la recherche *méthodique* des simplifications ; car la Géométrographie n'innove rien *en Géométrie*, elle se sert d'éléments connus, et, *à la rigueur*, toutes les simplifications auxquelles elle conduit, auraient pu théoriquement être faites sans elle ; mais on n'y songeait pas, parceque l'on manquait de criterium et parceque le mot de simplicité ne s'appliquait qu'à l'exposition théorique de la solution d'un problème et non à sa construction effectuée.

Voici l'expérience dont il s'agit. Je voulais prier un géomètre, tout à fait étranger à la notion de Géométrographie, de m'écrire en détail, *sans faire l'épure*, comment il s'y prendrait pour exécuter, *avec la règle et le compas*, le plus simplement qu'il le pourrait, selon les règles connues de la construction des expressions algébriques, une construction quelconque que je lui indiquerais au hasard. D'après cela je pourrais évaluer sa construction par le symbole géométrographique, puis reprendre, moi, le même problème en y apportant les simplifications méthodiques de la Géométrographie ; enfin évaluer ma construction et comparer les deux symboles. Je ne pouvais faire facilement la chose à Paris car les mathématiciens qui y sont de mes relations connaissent, au moins vaguement, la Géométrographie, et je n'aurai pas eu la construction dans les conditions qu'il me fallait. M. Jung avec qui j'ai le plaisir d'être en rapport, précisement à propos de Géométrographie, accepta de m'aider à faire l'expérience. Je lui envoyai alors cette construction :

Dans un triangle ABC dont les côtés sont a, b, c placer le point dont les distances aux trois côtés BC, CA, AB, (ou coordonnées normales), sont respectivement proportionelles à :

$$\frac{a^2b^2 + a^2c^2 - b^2c^2}{a}, \quad \frac{b^2c^2 + b^2a^2 - a^2c^2}{b}, \quad \frac{c^2a^2 + c^2b^2 - a^2b^2}{c}$$

c'est un point que je choisis au hasard parmi ceux qui se rencontrent fréquemment dans la Géométrie du triangle avec des coordonnées un peu complexes.

Il était convenu que le point devait être placé directement d'après ses coordonnées et non d'après des propriétés géométriques particulières à ce point, si l'on en trouvait. Monsieur Jung pria un de ses anciens élèves de faire ce que je désirais, et quelques jours après il eut la bonté de m'envoyer la rédaction même qui lui avait été remise.

Je dois dire que la construction qu'on y indique est très logique, fort bien conçue et analogue à celles que chacun de vous ferait sans doute, et que moi-même j'eusse fait il y a quelques années. On en trouverait, facilement, sans Géométrographie, de beaucoup plus simples à tracer, mais il n'y aurait aucun criterium pour permettre de l'affirmer, et cette préoccupation du tracé *réel* n'est pas dans l'esprit des Géomètres.

Je déterminai alors le symbole de cette construction que je conduisis *même assez économiquement* en évitant certaines répétitions de lignes inutiles. Puis je cherchai une construction de ce même point en appliquant les procédés géométrographiques et j'en déterminai aussi le symbole. Les résultats sont *stupéfiants* ; les voici :

Symbole de la construction qui m'a été envoyée

$$\text{op} : (81R_1 + 46R_2 + 211C_1 + 121C_3)$$

Simplicité 459 ; exactitude 292 ; 46 droites, 121 cercles.

Symbole de la construction géométrographique

$$\text{op} : (14R_1 + 7R_2 + 48C_1 + 22C_3)$$

Simplicité 91 ; exactitude 62 ; 7 droites, 22 cercles.

La première exige donc plus de 5 fois plus d'opérations élémentaires et le tracé de 6 fois $\frac{1}{2}$ plus de droites, de 5 fois $\frac{1}{2}$ plus de cercles !

Ce n'est pas tout, un géomètre qui s'est épris de la Géométrographie, s'y est exercé, (et y a même acquis beaucoup plus de dextérité que moi) Monsieur Bernès, auquel j'avais envoyé le problème, parceque je ne doutais pas qu'il n'en simplifiât encore la construction, m'écrit ce matin qu'il l'exécute avec 64 *opérations élémentaires* et qu'il me donnera la construction à son retour de voyage ! Plus de 7 fois plus simple que la construction faite en employant la méthode résultant de ce qui est indiqué dans tous les Traités classiques, au sujet des constructions. J'avoue que j'ai été moi-même un peu

surpris de cette invraisemblable différence, je pensais bien que je simplifierais d'au moins la moitié, mais de 5 fois, de 7 fois, je ne pouvais m'y attendre, d'autant plus que lorsque j'essayais moi-même d'exécuter les constructions d'application par les méthodes usuelles, pour faire la comparaison avec les méthodes géométrographiques, je les simplifiais malgré moi, pour éviter de les compliquer inutilement, et je n'arrivais pas alors à des résultats aussi éloignés dans les constructions traitées par moi avec les deux méthodes ; en m'adressant à un géomètre quelconque, j'ai évité cette cause d'incertitude, et, sans croire que *toutes* les constructions d'application—je ne parle pas des constructions fondamentales séculaires, que j'ai cependant presque *toutes* simplifiées peu ou beaucoup depuis : *mener par un point* A *une droite parallèle à une droite donnée* BC—donneraient lieu à une aussi considérable réduction, j'estime que la construction géométrographique serait *toujours* de 2 à 3 fois plus simple que la construction exécutée par les méthodes employées jusqu'ici.

Paris, *Octobre* 1894.

Note additionelle.

Si l'on convient que les longueurs doivent être comptées positivement à partir de o vers oA, on voit que AA_1 est égal à $-R_a$, l'origine des rayons étant en A. Les équations de la page 10 doivent donc être $X = \rho - R_a$, etc. Le signe du coefficient de ρ dans l'équation (1) sera simplement changé, et l'on aura les valeurs de ρ avec le signe qui leur convient.

The Nine-point Circle.

By R. F. Davis, M.A.

The nine-point circle of a triangle touches the inscribed circle.

Figure 2.

I. Let ABC be a triangle, having $\angle C$ greater than $\angle B$, D, E, F the middle points of the sides, and AX perpendicular to BC.

Then the upper segment of the nine-point circle cut off by DX contains an angle $C - B$; and conversely.

Figure 3.

II. If AP bisect the angle A and meet the base BC in P, and AC′ be taken along AB equal to AC; then PC′ touches the inscribed circle. Also $\angle BPC' = \angle C - \angle B$.

For the triangles APC, APC′ are congruent; hence the perpendiculars IM, IM′ on PC, PC′ respectively are equal.

Figure 4.

III. $$DM^2 = DP \cdot DX$$

For
$$\begin{aligned} HI^2 &= HC^2 \\ &= HD \cdot HK \\ &= HP \cdot HA; \end{aligned}$$

and the projections of HI, HP, HA on BC are DM, DP, DX.

Figure 5.

IV. Let O be a fixed point on the tangent at A to a fixed circle S, and points P, Q be taken (the one on OA and the other on OA produced) such that $OA^2 = OP \cdot OQ$, then the segment of a circle Σ, described through O, Q and containing an angle equal to the external angle between the tangents to S from P, touches the circle S.

For if PR, the second tangent to S from P, be drawn, and OR produced to meet S in T, since

$$OR \, . \, OT = OA^2 = OP \, . \, OQ \, ,$$

therefore $\angle OTQ = \angle OPR$;

therefore the point T lies on Σ.

Again, drawing the tangent TU to S at T to meet PR produced in U ;

$$\angle UTR = \angle URT = \angle ORP = \angle OQT \, ;$$

therefore TU touches Σ at T.

Thus the circles S, Σ touch each other in T.

V. The application of IV. is fairly obvious. Since in Figure 4,

$$DM^2 = DP \, . \, DX \text{ (III.)},$$

the segmental circle upon DX, containing an angle

$$BPC' = C - B \text{ (II.)},$$

touches the inscribed circle (V.). But (I.) the former circle is none other than the nine-point circle.

The Brocard Points and the Brocard Angle.

By R. F Davis, M.A.

Figure 6.

I. Construction for the Brocard points.

Let ABC be a triangle. Describe a circle touching AB in A and passing through C; draw the chord AP parallel to BC. Join BP meeting this circle in Ω.

Join $A\Omega$, $C\Omega$.

Then $$\angle \Omega AB = \angle \Omega CA, \quad = \angle \Omega PA \quad = \angle \Omega BC.$$

Similarly for Ω'.

II. Characteristic property of the Brocard angle.

Draw AX, PR perpendicular to BC.

Since AP, CQ are parallel chords, the triangles ACX, PQR are congruent by symmetry; therefore $$AX = PR, \quad CX = QR.$$

Now $$BR = BX + CX + CR$$
$$= BX + CX + QX;$$

therefore, dividing each of the terms by the equals AX or PR,

$$\cot\omega = \cot B + \cot C + \cot AQC$$
$$= \cot B + \cot C + \cot A.$$

On the Solitary Permanent Wave: A continuation.

By J. M'Cowan, M.A., D.Sc.

Second Meeting, December 14th, 1894.

JOHN M'COWAN, Esq., M.A., D.Sc., President, in the Chair.

Parabolic Note: Co-Normal Points.

By R. TUCKER, M.A.

1. If the coordinates of a point on a parabola,

$$y^2 - 4ax = 0,$$

be $(am^2, 2am)$, in which I call m the parameter, then the equations to the tangent and normal at the point are

$$x - my + am^2 = 0 \quad \ldots \quad \ldots \quad \text{(i.)}$$

and

$$mx + y - a(m^3 + 2m) = 0 \quad \ldots \quad \ldots \quad \text{(ii.)},$$

and to the chord through (m), (m') is

$$y(m + m') - 2x - 2amm' = 0 \quad \ldots \quad \ldots \quad \text{(iii.)}.$$

If we write (ii.) in the form

$$am^3 + (2a - x)m - y = 0 \quad \ldots \quad \ldots \quad \text{(iv.)},$$

we see that from a given point (x, y) we can draw three normals to the curve with the condition

$$\Sigma m = 0.$$

Let O be the point (x, y), and $P(m_1)$, $Q(m_2)$, $R(m_3)$ the corresponding points on the parabola: then I call these latter *co-normal* points, and the circle through them a *co-normal* circle.

2. We have

$$S_1 \equiv \Sigma m = 0,$$
$$S_2 \equiv \Sigma m^2 = -2\Sigma m_1 m_2,$$
$$S_3 = 3m_1 m_2 m_3 = 3\mu,$$
$$S_4 = S_2^2/2\,;$$

also

$$m_1^2 - m_2 m_3 = m_1^2 + m_1 m_2 + m_2^2 = \cdots = S_2/2.$$

3. In the case when P, Q, R are *any* three points on the curve the circle PQR is

$$x^2 + y^2 - ax[S_2 + \Sigma m_1 m_2 + 4] + ay[S_1 . \Sigma m_1 m_2 - \mu]/2 - a^2 \mu S_1 = 0 \quad \ldots \quad \text{(i.)}$$

and the tangent-circle pqr is

$$x^2 + y^2 - ax[1 + \Sigma m_1 m_2] - ay[S_1 - \mu] + a^2 \Sigma m_1 m_2 = 0, \quad \ldots \quad \text{(ii.)}$$

If the points are co-normal points, then these equations take the form

$$x^2+y^2-ax(S_2+8)/2-ay\mu/2=0, \quad \ldots \quad \ldots \text{(iii)}$$

$$x^2+y^2+ax(S_2-2)/2+ay\mu-a^2S_2/2=0 \ldots \quad \ldots \text{(iv.)}.$$

4. The co-ordinates of p (§ 3) for co-normal points, are

$$(am_2m_3, \ -am_1).^*$$

5. Through P, Q, R draw parallels to

(i.) the tangents at Q, R, P;

(ii.) „ R, P, Q;

and let P_q, Q_r, R_p; P_r, Q_p, R_q be the points where the sets (i.), (ii.) respectively meet the parabola (C.P. § 19). If PP_q, QQ_p meet in R_r, and in like manner for the other pairs, then R_r is given by $a(m_1^2+m_2^2-m_1m_2)$, $-am_3$. (C.P. § 24.)

Then the area of $P_pQ_qR_r$

$$=\pm\frac{a^2}{2}\begin{vmatrix} 1 & 1 & 1 \\ S_2/_2-2m_2m_3 & S_2/_2-2m_3m_1 & S_2/_2-2m_1m_2 \\ m_1 & m_2 & m_3 \end{vmatrix} = \text{area of } \triangle PQR;$$

and the equation to the circle is

$$x^2+y^2+(1-4S_2)ax/2+4\mu ay+3(S_2^2-S_2)a^2/4=0.$$

Again, since the coordinates of midpoint of Q_qR_r are

$$a(S_2/_2+m_1^2), \ am_1/2,$$

the N.P. circle of the triangle is given by

$$x^2+y^2-(6S_2+1)ax/4-2\mu ay+(4S_2^2+S_2)a^2/8=0.$$

6. If we draw Pr_1, Qr_1 parallel to the normals at Q, P, since Pr_1 is given by

$$y+m_2x=2am_1+am_1^2m_2,$$

* The first four articles in the text are taken from my paper, entitled *Some Properties of Co-normal Points on a Parabola* (*Proceedings of London Mathematical Society*, vol. xxi., pp. 442-451. Subsequent references are to the sections of this paper, *C.P.*

we see that r_1 is given by

$$-a(2+m_1m_2),\quad -a(2m_3+\mu),$$

hence $\Delta p_1q_1r_1 = \pm\frac{1}{2}a^2 \begin{vmatrix} 1 & 1 & 1 \\ 2+m_2m_3 & 2+m_3m_1 & 2+m_1m_2 \\ \mu+2m_1 & \mu+2m_2 & \mu+2m_3 \end{vmatrix} = \Delta\mathrm{PQR}$;

and since $p_1q_1 = \mathrm{PQ}$, the triangles are congruent.

The equation to $\mathrm{A}r_1$ is $y = m_3x$,
hence $\mathrm{A}r_1$ and the normal at R make equal angles with the axis.

The circle $p_1q_1r_1$ is given by

$$x^2+y^2-(8-\mathrm{S}_2/2)ax+5a\mu y/2+3a^2(\mu^2-2\mathrm{S}_2+8)/2=0\,;$$

and the N.P. circle is given by

$$x^2+y^2+(2-\mathrm{S}_2/4)ax+7a\mu y/4-a^2(2\mathrm{S}_2-3\mu^2)/4=0.$$

7. The equations to PP_p, QQ_q (§ 6) are

$$m_1x-m_2m_3y=am_1(m_1^2-2m_2m_3),$$
$$m_2x-m_3m_1y=am_2(m_2^2-2m_3m_1),$$

hence they intersect in r_2, on the ordinate of R, given by

$$am_3^2,\quad -am_1m_2/m_3.$$

Similarly for the analogous points p_2, q_2.

Hence $$\Delta p_2q_2r_2 = \tfrac{1}{2}\,\Delta\mathrm{PQR}.$$

The circle $p_2q_2r_2$ is given by

$$x^2+y^2-(\mathrm{S}_2+1)ax+(\mu+\mathrm{S}_2^2/4\mu)ay+a^2(4\mathrm{S}_2+\mathrm{S}_2^2)/4=0. \quad \ldots \quad \text{(i.)}$$

The points $p_2q_2r_2$, lie on the rectangular hyperbola

$$xy = -\mu a^2 \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii.)}$$

which cuts (i.) again in $(a,\ -a\mu)$.

The equation to the perpendicular from r_2 on p_2q_2 is

$$m_3y+m_1m_2x=am_1m_2(m_3^2-1),$$

hence the orthocentre of $p_2q_2r_2$ is $(-a,\ a\mu)$.

This point is on (ii) and *coincides with the orthocentre* of pqr (C.P. §13).

The N.P. circle of $p_2q_2r_2$ is the co-normal circle

$$x^2+y^2+ax(1-S_2)/2+ay(S_2^2-4\mu^2)/8\mu=0.$$

The radical axis of this circle and of PQR is

$$36\mu^2x+S_2^2y=0.$$

The tangent from the focus to (i.) is $aS_2/2$.

The equation to pp_2 is

$$S_2x-6\mu y=2a(m_1^4-m_1\mu+3m_2^2m_3^2),$$

hence pp_2, qq_2, rr_2 are parallel.

Also the equation to p_2q_2 is

$$m_1m_2y-m_3x=-a(m_1^2+m_2^2)m_3,$$

i.e., the line is parallel to Rr.

The points p, q, r lie on the hyperbola (ii.): hence we see otherwise that the orthocentres of pqr, $p_2q_2r_2$ coincide, and that the two circles cut the Latus Rectum in the same point $(a, -a\mu)$, the join of which with the common orthocentre is a diameter of the hyperbola.

8. The orthocentre of $p_1q_1r_1$ (§6) being $(2a, -a\mu/2)$ is on the hyperbola (§7, ii.). See C.P. §12.

From §11 of C.P. we see that the centre of perspective of the triangles PQR, pqr, viz. $(aS_2/6, -6a\mu/S_2)$ is also on the same curve.

9. If the sides QR, RP, PQ produced cut the diameters through P, Q, R in L, M, N these points are given by

$$\text{L}, [a(m_1^2+m_2m_3), 2am_1], \text{ etc.};$$

hence $$\triangle\text{LMN}=2\triangle\text{PQR}.$$

The circle LMN has for its equation

$$x^2+y^2-2ax-2a\mu y-a^2(S_2^2+4S_2)/4=0.$$

The orthocentre of LMN is

$$a(S_2-4)/2, \ -2a\mu.$$

If n is the midpoint of LM, it is given by

$$(-am_1m_2, \ -am_3)$$

and therefore it and the analogous points l, m, lie on the rectangular hyperbola $$xy=\mu a^2. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(i.)}$$

From the above we see that pl, qm, rn are diameters of the parabola, and lm, pq; mn, qr; nl, rp; intersect on the tangent at the vertex and are isoclinals to it.

The equation to the circle lmn is

$$x^2+y^2+ax(2-S_2)/2+a\mu y-a^2S_2/2=0\,;$$

and it is therefore § 3 (iv.) equal to the circle pqr.

The perpendiculars from l, m, n on QR, RP, PQ respectively meet in $(2a, a\mu/2)$, which is on (i.); and the perpendiculars from P, Q, R on mn, nl, lm meet in $[a(S_2-4)/2, \ -a\mu]$, *i.e.* O′ of C.P. § 15.

10. If the join of P to the midpoint of QR cuts the parabola in p_3, the parameter of this point is $-S_2/3m_1$, hence the corresponding tangent circle of the triangle $p_3q_3r_3$ is given by

$$x^2+y^2-ax-ayS_2^2(9+2S_2)/54\mu=0.$$

The vertices of this tangent triangle are

$$(aS_2^2/9m_1m_2, \ am_3S_2/3m_1m_2)$$

so that its centroid is $\quad (0, \ -S_2^2/18\mu)$;

and its orthocentre $\quad (-a, \ 7aS_2^3/54\mu)$.

11. Through P, Q, R draw lines parallel to QR, RP, PQ respectively, these lines meet the parabola in the co-normal points, whose parameters are $-2m_1$, $-2m_2$, $-2m_3$; and the lines cut one another in

$$(-2am_2m_3, \ -4am_1), \ (-2am_3m_1, \ -4am_2), \ (-2am_2m_3, \ -4am_3).$$

Take the images of these points in the vertex, viz $(2am_1m_3, 4am_1)$, etc., and we find its circumcircle to be given by

$$x^2+y^2-(8-S_2)ax-a\mu y-8S_2a^2=0,$$

the centre of which is the orthocentre of PQR (C.P. § 13.)

12. The lines QR, AP cut in $p_a\,(-am_2m_3/2,\ -am_2m_3/m_1)$, RP, AQ in q_a, and PQ, AR in r_a; hence $p_aq_ar_a$, which is the *central* triangle of the quadrilateral APQR,

$$=\tfrac{1}{2}\Delta PQR.$$

The circle $p_aq_ar_a$ has for its equation

$$x^2+y^2+2ax+ay(S_2^2/\mu^2)/4\mu+a^2S_2/2=0.$$

The equation to p_aq_a is

$$m_1m_2y+2m_3x=+a\mu.$$

13. If (cf. C.P. §29) we draw lines from P, Q, R through the point, $x=ka$ on the axis to cut the curve in T_1', T_2', T_3', then as T_1' is given by $(-k/m_1)$ the equation to the tangent-circle for $T_1'T_2'T_3'$ will differ from that to the tangent-circle for $T_1T_2T_3$ only in the sign of k, *i.e.*, it will be

$$x^2+y^2-ax-ay(k\,.\,S_2+2k^3)/2\mu=0.$$

14. If through p, q, r we draw the corresponding diameters, the vertices of these diameters are co-normal points, viz.,

$$(am_1^2/4,\ -am_1),\ \text{etc.},$$

and the co-normal circle through the vertices is

$$x^2+y^2-ax(S_2+32)/8+ay\mu/16=0.$$

15. Parallels through p to QR and through q parallel to RP intersect on the diameter through r.

16. Parallels through P to AQ, AR, meet the parabola in points whose parameters are

$$(m_2-m_1),\ (m_3-m_1),$$

hence we get two sets of co-normal points.

The equations to the co-normal circles are

$$x^2+y^2-ax(3S_2+8)/2 \mp ayk/2=0,$$

where $$k \equiv m_2-m_1 \,.\, m_3-m_1 \,.\, m_1-m_2.$$

17. The median of PQR which passes through P cuts the parabola in the point whose parameter is $(-S_2/3m_1)$, hence the corresponding tangent-circle has for its equation

$$x^2+y^2-ax-ayS_2^2(9+2S_2)/54\mu=0.$$

18. If in § 12 q_a, r_a are outside the curve, then the midpoints of QR, AP, $q_a r_a$ are given by

$$a(m_2^2+m_3^2)/2,\ -am_1\ ;\ \ am_1^2/2,\ am_1\ ;\ \ am_1^2/4,\ -am_1(m_2^2+m_3^2)/2m_2m_3\ ;$$

hence the *central* axis of APQR is

$$-2m_2m_3y+4m_1x=m_1S_2a.$$

19. The poles of the co-normal chords are

$$-a(m_1^2+2),\ -2a/m_1\ ;\ \ -a(m_2^2+2),\ -2a/m_2\ ;\ \ -a(m_3^2+2),\ -2a/m_3.$$

These poles lie upon the line

$$\mu y-2x=a(S_2+4). \qquad \text{(cf. C.P. § 17.)}$$

The diameters through the poles meet the curve in

$$a/m_1^2,\ -2a/m_1\ ;\ \text{etc.}\ ;$$

hence the circle through the vertices of these diameters is

$$x^2+y^2-ax(S_2^2+4\mu^2)/\mu^2+ay/2\mu+a^2S_2/2\mu^2=0\ ;$$

and the corresponding tangent-circle is

$$x^2+y^2-ax-ay(S_2+2)/2\mu=0.$$

The sides of this last triangle are

$$m_1y-2m_2m_3x=2a, \quad \text{etc.},$$

∴ the perpendiculars are

$$m_1^2x+2\mu y=a(1-4m_2m_3), \quad \text{etc.},$$

whence the orthocentre is

$$[-4a,\ a(1+2S_2)/2\mu].$$

Geometrical Problem.

By G. E. Crawford, M A.

Figure 24.

Let OQ, OR be two straight lines meeting at O, and P any point. Required to draw through P a straight line cutting off a given area OAB from the two straight lines.

Draw PD parallel to OR cutting OQ in D.

Construct a $\triangle$OPC equal to the given area, and such that OP is one of its sides, and that another of its sides, OC, lies along OQ.

Take OE a mean proportional to OC, OD.

Draw OF perpendicular to OC and equal to half of it.

Join EF, and cut off FG = OF.

Take OA = EG. Then PAB is the required straight line.

Proof :

Sqs. on OE, OF = sq. on EF

= sqs. on EG, GF, + 2 rect. EG . GF

$\therefore$ sq. on OE = sq. on EG + 2 rect. EG . GF

= sq. on OA + rect. OA . OC (since OC = 2GF)

$\therefore$ rect. OC . OD = sq. on OA + rect. OA . OC

rect. OC . (OA + AD) = sq. on OA + rect. OA . OC

$\therefore$ rect. OC . OA + OC . DA = sq. on OA + rect. OA . OC

$\therefore$ sq. on OA = rect. OC . DA

$\therefore$ OC : DA :: $OA^2 : DA^2$

$\therefore$ $\triangle OPC : \triangle DPA :: \triangle OAB : \triangle DAP$

$\therefore$ $\triangle OAB = \triangle OPC$ = given area.

Colour-sensation and Colour-blindness, with Experiments.

By Wm. Peddie, D.Sc.

Third Meeting, January 11th, 1895.

WM. PEDDIE, Esq., D.Sc., F.R.S.E., Vice-President, in the Chair.

Properties connected with the Angular Bisectors of a Triangle.

By J. S. MACKAY, M.A., LL.D.

NOTATION.

When points and lines are not specifically designated in the course of the following pages it will be understood that the notation for them is that recommended in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I. pp. 6–11 (1894). It may be convenient to repeat all that is necessary for the present purpose.

A' B' C' = mid points of the sides BC CA AB

D E F = points of contact of sides with incircle

D_1 E_1 F_1 = " " " " " " first excircle.

And so on.

H = orthocentre of ABC

I = incentre of ABC

I_1 I_2 I_3 = 1st 2nd 3rd excentres of ABC

L M N = feet of interior angular bisectors of ABC

L' M' N' = " " exterior " " " "

O = circumcentre of ABC

U U' = ends of that diameter of the circumcircle which is perpendicular to BC. U is on the opposite side of BC from A.

Similarly for V V' and W W'.

X Y Z = feet of the perpendiculars from A B C.

The various points

$$K\,K', \quad P\,P', \quad Q\,Q', \quad S\,S', \quad T\,T'$$

are defined as they occur.

α	β	γ	$=$ AI	BI	CI	
α_1	β_1	γ_1	$=$ AI_1	BI_1	CI_1	
α_2	β_2	γ_2	$=$ AI_2	BI_2	CI_2	
α_3	β_3	γ_3	$=$ AI_3	BI_3	CI_3	
$\alpha_1-\alpha$	$\beta_2-\beta$	$\gamma_3-\gamma$	$=$ I_1I	I_2I	I_3I	
$\alpha_2+\alpha_3$	$\beta_3+\beta_1$	$\gamma_1+\gamma_2$	$=$ I_2I_3	I_3I_1	I_1I_2	
h_1	h_2	h_3	= the perpendiculars AX BY CZ			
l_1	l_2	l_3	= the interior angular bisectors of A B C			
λ_1	λ_2	λ_3	= ,, exterior ,, ,, ,, ,, ,, ,,			
	r		= radius of the incircle			
r_1	r_2	r_3	= radii of the 1st 2nd 3rd excircles			
	R		= radius of the circumcircle			
	s		= semiperimeter of ABC			
s_1	s_2	s_3	$= s-a$	$s-b$	$s-c$	
u_1	v_1	w_1	= BL	CM	AN	
u_1'	v_1'	w_1'	= BL′	CM′	AN′	
u_2	v_2	w_2	= CL	AM	BN	
u_2'	v_2'	w_2'	= CL′	AM′	BN′	

§ 1.

If from either end of the base of a triangle a perpendicular be drawn to the bisector of the interior or exterior vertical angle, the distance of the foot of this perpendicular from the mid point of the base is equal to half the difference or half the sum of the sides of the triangle.*

FIGURE 7.

Let BP BP′, the perpendiculars from B on AL AL′, the interior and the exterior bisectors of ∠ A, meet AC in B_1 B_2.

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* Compare Leybourn's *Mathematical Repository*, old series, I. 284 (1799), II. 24 (1801).

Then $\quad BP = B_1P \qquad BP' = B_2P'$

and $\quad CB_1 = AC - AB \qquad CB_2 = AC + AB.$

Now since $P\ P'$ are the mid points of $BB_1\ BB_2$ and A' the mid point of BC,

therefore
$$A'P = \tfrac{1}{2}CB_1 \qquad A'P' = \tfrac{1}{2}CB_2$$
$$= \tfrac{1}{2}(AC - AB) \qquad = \tfrac{1}{2}(AC + AB)$$

Similarly if $Q\ Q'$ be the feet of the perpendiculars from C on $AL\ AL'$,
$$A'Q = \tfrac{1}{2}(AC - AB) \qquad A'Q' = \tfrac{1}{2}(AC + AB).$$

It is not easy to assign authorities to the properties given in the following pages. Some of these properties occur incidentally in the solutions of problems on the construction of triangles, and are there spoken of, or assumed without being spoken of, as well known theorems. A large collection of them will be found in four articles entitled "Useful Propositions in Geometry" by M. A. Harrison, which appeared in Leybourn's *Mathematical Repository*, old series, I. 283-5, 367-9, II. 23-5, 234-7 (1799-1801). In these articles no mention is made of properties connected with the bisector of the exterior vertical angle.

It has been conjectured that "M. A. Harrison" is a pseudonym, adopted either by J. H. Swale or John Lowry.

(1) $\angle ABB_1 = AB_1B = BAL' = B_2AL' = AP'A' = AQ'A'$
$$= \tfrac{1}{2}(B + C)$$
$$\angle CBB_1 = BL'A = \tfrac{1}{2}(B - C)$$

(2) $A'PP'$ is a straight line parallel to AC. Hence $P\ P'$ are situated on $C'A'$ one of the sides of the triangle $A'B'C'$, which is complementary to ABC.

Similarly, if from B perpendiculars be drawn to the bisectors of the interior and exterior angles at C, the feet of these perpendiculars will also be situated* on $C'A'$.

(3) If perpendiculars be drawn from each vertex of a triangle to the interior and the exterior bisectors of the angles at the other vertices, the twelve points of intersection thus obtained will range, four

* Arthur Lascases in the *Nouvelles Annales*, XVIII. 171 (1859).

and four, on three straight lines, which by their mutual intersections will form the triangle complementary to the given triangle. *

The proof of this is obvious enough from what precedes; but the following demonstration will be found interesting.

FIGURE 8.

Let ABC be a triangle, $I\ I_1\ I_2\ I_3$ the incentre and the ex-centres.

The four lines $I_2B\ I_3I_1\ I_3C\ I_1I_2$ are the interior and the exterior bisectors of the angles B and C. Now these four lines, taken three and three, form the four triangles

$$I_3I_1C \quad I_2IC \quad I_2I_1B \quad I_3IB$$

Hence, by a theorem due to Wallace,† the circumcircles of these four triangles all pass through the same point A; and by one of Steiner's theorems ‡ the feet of the perpendiculars let fall from A on the four straight lines are collinear.

Let $A_1\ A_2\ A_3\ A_4$ be the feet of the perpendiculars.
Then AA_1BA_2 is a rectangle;
therefore A_1A_2 passes through C′ the mid point of AB.
Similarly A_3A_4 „ „ B′ „ „ „ „ AC;
therefore the straight line $A_1A_2A_3A_4$ bisects AB and AC.

(4)
$$A_3A_4 = b, \quad A_1A_2 = c, \quad A_2A_4 = s$$
$$A_1A_3 = s_1, \quad A_2A_3 = s_2, \quad A_1A_4 = s_3$$

(5) The four points $A_1\ A_2\ A_3\ A_4$ lie, two and two, on the circumferences of the six circles which have for diameters the distances of A from $I\ I_1\ I_2\ I_3\ B\ C$.

(6) If the circles be denoted by their diameters, the circles $AI\ AI_1$ touch each other at A, and have I_2I_3 for common tangent; they also touch the circle II_1 the former at I and the latter at I_1.

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* T. T. Wilkinson in the *Lady's and Gentleman's Diary* for 1862, p. 74. The demonstration given is also due to him, as well as part of (4). See the *Diary* for 1863, pp. 54-5.

† Leybourn's *Mathematical Repository*, new series, Vol. I., p. 22 of the Questions (1804).

‡ Gergonne's *Annales* XVIII. 302 (1828) or Steiner's *Gesammelte Werke*, I. 223

The circles AI_2 AI_3 touch each other at A, and have AI_1 for common tangent; they also touch the circle $I_2 I_3$ the former at I_2 and the latter at I_3.

(7) The radical axis of the circles

AB	AI	AI_2	is	AA_1
AB	AI_3	AI_1	,,	AA_2
AC	AI	AI_3	,,	AA_3
AC	AI_1	AI_2	,,	AA_4

(8)
$$AP : AL = \tfrac{1}{2}(AC + AB) : AC$$
$$AQ : AL = \tfrac{1}{2}(AC + AB) : AB$$

Figure 9.

Since PA' is parallel to AC,

therefore triangles ACL $PA'L$ are similar;

therefore
$$AL : PL = AC : PA'$$
$$= AC : \tfrac{1}{2}(AC - AB)$$

therefore
$$AL - PL : AL = \tfrac{1}{2}(AC + AB) : AC$$

(9)
$$AP' : AL' = \tfrac{1}{2}(AC - AB) : AC$$
$$AQ' : AL' = \tfrac{1}{2}(AC - AB) : AB$$

(10)
$$PQ : AL = AC^2 - AB^2 : 2AC \cdot AB$$
$$P'Q' : AL' = AC^2 - AB^2 : 2AC \cdot AB$$

Figure 9.

Since triangles ACL $PA'L$ are similar

therefore
$$PL : AL = PA' : AC$$
$$= \tfrac{1}{2}(AC - AB) : AC$$

Since triangles ABL $QA'L$ are similar

therefore
$$QL : AL = QA' : AB$$
$$= \tfrac{1}{2}(AC - AB) : AB$$

therefore $$\frac{PL+QL}{AL}=\frac{AC-AB}{2AC}+\frac{AC-AB}{2AB}$$

$$=\frac{AC\cdot AB-AB^2}{2AC\cdot AB}+\frac{AC^2-AC\cdot AB}{2AC\cdot AB}$$

$$=\frac{AC^2-AB^2}{2AC\cdot AB}$$

(11)* $$ABC=AQ\cdot BP=AP\cdot CQ$$
$$=AQ'\cdot BP'=AP'\cdot CQ'$$

For triangles AXL, BPL are similar

therefore $$AX:BP=AL:BL$$
$$=QL:A'L$$
$$=AQ:BA'$$

therefore $$AQ\cdot BP=AX\cdot BA'$$
$$=ABC$$

The last two expressions for ABC may be derived from the first two, since

$$AP=BP' \quad AQ=CQ' \quad BP=AP' \quad CQ=AQ'.$$

(12) The values of the following angles may be registered for reference:

$$ABP=ACQ=90°-\tfrac{1}{2}A$$
$$BDF=BFD=BD_2F_2=BF_2D_2=90°-\tfrac{1}{2}B$$
$$CDE=CED=CD_3E_3=CE_3D_3=90°-\tfrac{1}{2}C$$

$$ABP'=ACQ'=\tfrac{1}{2}A$$
$$BD_1F_1=BF_1D_1=BD_3F_3=BF_3D_3=\tfrac{1}{2}B$$
$$CD_1E_1=CE_1D_1=CD_2E_2=CE_2D_2=\tfrac{1}{2}C$$

In triangles BDP CQD

$$PBD=\tfrac{1}{2}(B-C),\quad BDP=90°+\tfrac{1}{2}C,\quad DPB=90°-\tfrac{1}{2}B$$
$$=DCQ \qquad\qquad =CQD \qquad\qquad =QDC$$

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* Part of (11) is given in Hind's *Trigonometry*, 4th ed., p. 304 (1841).

In triangles BD_1P CQD_1

$$PBD_1 = \tfrac{1}{2}(B - C), \quad BD_1P = \tfrac{1}{2}C, \quad D_1PB = 180^\circ - \tfrac{1}{2}B$$
$$= D_1CQ \qquad = CQD_1 \qquad = QD_1C$$

In triangles BD_2P' $CQ'D_2$

$$P'BD_2 = \tfrac{1}{2}A + B, \quad BD_2P' = \tfrac{1}{2}C, \quad D_2P'B = 90^\circ - \tfrac{1}{2}B$$
$$= D_2CQ' \qquad = CQ'D_2 \qquad = Q'D_2C$$

In triangles BD_3P' $CQ'D_3$

$$P'BD_3 = \tfrac{1}{2}A + C, \quad BD_3P' = 90^\circ - \tfrac{1}{2}C, \quad D_3P'B = \tfrac{1}{2}B$$
$$= D_3CQ' \qquad = CQ'D_3 \qquad = Q'D_3C$$

$$AEP = AFP = AQE_1 = AQF_1 = 90^\circ + \tfrac{1}{2}C$$
$$APE = APF = AE_1Q = AF_1Q = \tfrac{1}{2}B$$

$$AE_2P' = AF_2P' = AQ'E_3 = AQ'F_3 = \tfrac{1}{2}C$$
$$AP'E_2 = AP'F_2 = AE_3Q' = AF_3Q' = \tfrac{1}{2}B$$

(13)* $$AP \cdot AQ = BP' \cdot CQ' = s\,s_1$$
$$BP \cdot CQ = AP' \cdot AQ' = s_2 s_3$$

The similar triangles AEP AQE_1 give

$$AE : AP = AQ : AE_1$$

therefore $$AP \cdot AQ = AE \cdot AE_1$$
$$= ss_1 \quad ;$$

and $$AP = BP' \quad AQ = CQ'.$$

The other equalities may be derived from the similar triangles BDP CQD, and the fact that

$$BP = AP' \quad CQ = AQ'.$$

(14) $$AP \cdot AQ \cdot BP \cdot CQ = AP' \cdot AQ' \cdot BP' \cdot CQ'$$
$$= \Delta^2$$

* (13) Half of this is given in Hind's *Trigonometry*, 4th ed., p. 304 (1841).

(15) Let D, D_1, D_2, D_3 be the points where the incircles and the excircles touch BC.

FIGURE 9.

It is known that

$$A'D = A'D_1 = \tfrac{1}{2}(AC - AB), \quad A'D_2 = A'D_3 = \tfrac{1}{2}(AC + AB);$$

hence $D \ D_1 \ P \ Q$ lie on a circle with centre A'

and $D_2 \ D_3 \ P' \ Q'$ „ „ „ „ „ „ „

(16) The incircle and first excircle of ABC cut the circle DPD_1Q orthogonally, and the second and third excircles cut $D_2Q'P'D_3$ orthogonally.

For $D\,D_1$ is perpendicular to $I\,D$ and I_1D_1;
and D_2D_3 „ „ „ I_2D_2 „ I_3D_3.

(17)
$$I\,P \cdot I\,Q = r^2, \quad I_1P \,.\, I_1Q = r_1^2$$
$$I_2P' \cdot I_2Q' = r_2^2, \quad I_3P' \cdot I_3Q' = r_3^2$$

(18) If $I \ I_1$ be considered as one pair of a system of coaxal circles, then $P \ Q$ are the limiting points of the system; and $P' \ Q'$ are the limiting points of the coaxal system of which $I_2 \ I_3$ form one pair.

For DD_1 is a common tangent to the circles $I \ I_1$, and A' is its mid point; therefore the radical axis of $I \ I_1$ passes through A'.

Now since the circle whose diameter is DD_1 has its centre at A' and cuts $I \ I_1$ orthogonally, therefore it passes through the limiting points of the system $I \ I_1$; and the limiting points of the system $I \ I_1$ are situated on the line II_1.

(19) $APBP'$ is a rectangle, and PP' bisects AB. Hence if AX be perpendicular to BC, the circle on AB as diameter passes through* $P \ P' \ X$.

Similarly the circle on AC as diameter* passes through $Q \ Q' \ X$.

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* W. H. Levy in the *Lady's and Gentleman's Diary* for 1856, p. 49.

(20) Triangles XPQ, XP′Q′ are inversely similar* to ABC.

FIGURE 9.

Since A P B X are concyclic

therefore $\angle APX = \angle ABX$

therefore $\angle XPQ = \angle ABC$

Since A C Q X are concyclic

therefore $\angle AQX = \angle ACX$

therefore triangle XPQ is similar to ABC

In like manner XP′Q′ is similar to ABC

(21) The directly similar triangles XPQ XP′Q′ have their homologous sides mutually perpendicular.

(22) The incentre and the excentres of triangles XPQ XP′Q′ are situated on BX and AX.

Since A P B X are concyclic

therefore $\angle BXP = \angle BAP = \frac{1}{2}A$

Since A C Q X are concyclic

therefore $\angle CXQ = \angle CAQ = \frac{1}{2}A$

therefore BX bisects $\angle PXQ$

therefore BX contains the incentre and one excentre of XPQ.

Now AX is perpendicular to BX

therefore AX contains the other excentres.

In like manner it may be proved that AX contains the incentre and one excentre of triangle XP′Q′, and that BX contains the other excentres.

* W. H. Levy in the *Lady's and Gentleman's Diary* for 1856, p. 49. The first part of the theorem, however, is given in Leybourn's ***Mathematical Repository***, old series, II. 25 (1801).

(23) To determine the incentre and the excentres of the triangles XPQ XP′Q′.

Since AX ID are parallel

therefore $$AL : IL = XL : DL.$$

But in the similar triangles ABC XPQ
AL and XL are homologous lines;
therefore IL and DL are homologous lines,
and I D homologous points;
therefore D is the incentre of XPQ.

Since $\angle DPD_1$ is right, D_1 is the first excentre.

The other excentres are the points where DP and DQ intersect AX.

In like manner it may be proved that D_3 and D_2 are the third and second excentres of triangle XP′Q′ and that the incentre and first excentre are the points where D_3Q' and D_3P' intersect AX.

(24) The circumcircles* of XPQ XP′Q′ pass through A′.

Since $$\angle A'XQ = \angle CAQ$$
$$= \angle A'PQ$$

because A′P and CA are parallel;
therefore A′ P X Q are concyclic.

In like manner A′ Q′ P′ X are concyclic.

(25) The diameters* of the circles XPQ XP′Q′ are respectively equal to AU′ AU.

For the diameter of XPQ is the perpendicular drawn from A′ to PQ and terminated by AX; and this perpendicular along with A′U′ U′A AX produced forms a parallelogram.

(26) The diameters of the circles XPQ XP′Q′ coincide with the radical axes of the circles I I_1 and I_2 I_3.

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* The first parts of (24) and (25) are found in Leybourn's *Mathematical Repository*, old series, II. 24, 235 (1801).

(27) The circle XPQ cuts orthogonally the system of circles I I_1; and the circle XP′Q′ cuts orthogonally the system I_2 I_3.

For the circle XPQ passes through the limiting points P Q of the system I I_1 and has its centre on the radical axis of the same system.

(28) The centres of the circles XPQ and XP′Q′ and the nine-point centre of triangle ABC are collinear.

For they are situated on the straight line which bisects A′X perpendicularly.

(29) The sum of the areas of the circles XPQ XP′Q′ is equal to the area of the circle ABC.

For the areas of circles are proportional to the squares of their diameters
and

$$AU'^2 + AU^2 = UU'^2.$$

(30) $$XPQ + XP'Q' = ABC.^*$$

(31)† $$ABC : XPQ = UU' : U'K$$
$$ABC : XP'Q' = UU' : UK$$

For $$ABC : XPQ = UU'^2 : AU'$$
$$= UU' : U'K$$

For another proof see §4, (11).

(32) $$XP \quad XP' \quad XQ \quad XQ'$$

are respectively parallel to

$$CU \quad CU' \quad BU \quad BU'$$

For $$\angle XPQ = \angle ABC$$
$$= \angle AUC.$$

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* W. H. Levy in the *Lady's and Gentleman's Diary* for 1855, p. 71.

† Parts of (28), (29), (30), (31), are found in Leybourn's *Mathematical Repository*, old series, II. 236, 25 (1801).

(3) The triangles

$$A'BP \quad A'CQ \quad A'BP' \quad A'CQ'$$

are respectively similar to

$$QAX \quad PAX \quad Q'AX \quad P'AX\,.$$

For $A'P$ is parallel to CA;

therefore $$\angle BA'P = C$$
$$= \angle AQX\,;$$

and $$\angle BPA' = 90^\circ + \tfrac{1}{2}A$$
$$= \angle AXQ.$$

Or it may be proved that

$$\angle A'BP = \angle QAX$$

since the sides of the one are perpendicular to the sides of the other.

(34) The triangles

$$A'UP \quad A'UQ \quad A'U'P' \quad A'U'Q'$$

are respectively similar to

$$QCX \quad PBX \quad Q'CX \quad P'BX.$$

For $$\angle A'UP = \angle QCX$$

since the sides of the one are perpendicular to the sides of the other;

and $$\angle A'PU = \tfrac{1}{2}A$$
$$= \angle QXC.$$

(35) The following triads of points are collinear:*

$$P\ D\ E\,;\quad P\ D_1\ E_1\,;\quad P'\ D_2\ E_2\,;\quad P'\ D_3\ E_3$$
$$Q\ D\ F\,;\quad Q\ D_1\ F_1\,;\quad Q'\ D_2\ F_2\,;\quad Q'\ D_3\ F_3.$$

---

* W. H. Levy in the *Lady's and Gentleman's Diary* for 1857, p. 51.

FIGURE 10.

The points B D P I are concyclic ;
therefore $\angle$ BDP is the supplement of $\angle$ BIP.
Because the isosceles triangles CDE, UBI have

$$\angle C = \angle U$$

therefore $\qquad \angle CDE = \angle BIU$;
therefore $\angle$ BDP is the supplement of $\angle$ CDE ;
therefore DP coincides with DE.

(36) The following quintets of points are concyclic :

$$\begin{array}{lllll} B & D & I & F & P; \\ B & D_1 & I_1 & F_1 & P; \\ B & D_2 & I_2 & F_2 & P'; \\ B & D_3 & I_3 & F_3 & P'; \end{array} \qquad \begin{array}{lllll} C & D & I & E & Q \\ C & D_1 & I_1 & E_1 & Q \\ C & D_2 & I_2 & E_2 & Q' \\ C & D_3 & I_3 & E_3 & Q' \end{array}$$

the diameters of the various circles being

$$\begin{array}{llll} BI & BI_1 & BI_2 & BI_3 \\ CI & CI_1 & CI_2 & CI_3 \end{array}$$

(37) Since C L B L′ form a harmonic range,
and CQ BP AL′ are parallel,
therefore Q L P A form a harmonic range.*

FIGURE 9.

Similarly for Q′ A P′ L′.

(38) If at L a perpendicular be drawn to AL meeting AB AC at P Q, then AP or AQ is a harmonic mean† between AB AC.

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* Fuhrmann's *Synthetische Beweise planimetrischer Sätze*, pp. 58-9 (1890).
† Rev. R. Townsend in *Mathematical Questions from the Educational Times*, XIV. 76 (1870).

FIGURE 11.

If B_1 be the image of B in AL,
then AL bisects $\angle BLB_1$;
therefore LQ bisects $\angle CLB_1$;
therefore $L(BAB_1Q)$ is a harmonic pencil.
Now this pencil is cut by the transversal AC;
therefore $A\ B_1\ Q\ C$ form a harmonic range
and AQ is the harmonic mean between $AB_1\ AC$.

Similarly for L'.

(39) $$AU : IU = AB + AC : BC$$

FIGURE 12.

Draw $IP\ IQ$ parallel to $AB\ AC$.

The quadrilaterals $ABUC\ IPUQ$ are similar;
therefore $$AU : IU = AB + AC : IP + IQ.$$

Now $$\angle UBL = \angle UAC = \angle UAB = \angle UIP;$$
and $$UB = UI;$$
therefore triangles $UBL\ UIP$ are congruent,
and $$BL = IP.$$
Similarly $$CL = IQ;$$
therefore $$AU : IU = AB + AC : BL + CL.$$

(40) If on $A'K$ as diameter a circle is described, and from O the circumcentre of ABC a perpendicular is drawn to $A'K$ meeting this circle at P, then*

$$AB^2 + AC^2 = 4PU^2.$$

* John Whitley in the *Gentleman's Mathematical Companion* for 1803, p. 38.

FIGURE 13.

The triangle PUU' is isosceles;

therefore $$PA'^2 = PU^2 - UA' \cdot A'U' = PU^2 - A'B^2;$$

and $$PK^2 = PU^2 - UK \cdot KU' = PU^2 - AK^2.$$

Now $$\begin{aligned} AB^2 + AC^2 &= 2A'B^2 + 2A'A^2 \\ &= 2A'B^2 + 2AK^2 + 2A'K^2 \\ &= 2A'B^2 + 2AK^2 + 2PA'^2 + 2PK^2 \\ &= 4PU^2 \end{aligned}$$

§ 2.

If from the mid point of the base of a triangle a perpendicular be drawn to the bisector of the interior or exterior vertical angle, this perpendicular will cut off from the sides segments equal to half the sum* or half the difference of the sides.

FIGURE 14.

Let the perpendiculars from A' to AU AU' meet AC at S S', and AB at T T'.

Draw BB_1 BB_2 parallel to the perpendiculars.

Because A' is the mid point of BC,

therefore S ,, ,, ,, ,, ,, B_1C.

Now $B_1C = AC - AB$;

therefore $CS = \frac{1}{2}(AC - AB)$;

therefore $AS = \frac{1}{2}(AC + AB)$.

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* Part of this is found in Leybourn's *Mathematical Repository*, old series, I. 284 (1799).

Similarly $\quad BT = AT' = AS' = \frac{1}{2}(AC - AB)$

and $\quad AT = BT' = CS' = \frac{1}{2}(AC + AB)$

(1) $\quad \angle ATS = \angle AST = \angle ABB_1 = \frac{1}{2}(B + C)$

$\quad \angle BA'T = \angle CA'S = \angle CBB_1 = \frac{1}{2}(B - C)$

(2) $\quad AS^2 + CS^2 = AT^2 + BT^2 = \frac{1}{2}(b^2 + c^2)$

(3) $\quad AS^2 - CS^2 = AT^2 - BT^2 = bc$

(4) $\quad AS \cdot CS = AT \cdot BT = \frac{1}{4}(AC^2 - AB^2)$

$\quad = \frac{1}{4}(CX^2 - BX^2) = A'B \cdot A'X$

(5) $\quad AS : CS = AT : BT = b + c : b - c$

(6) $\quad SS' = AB = c, \quad TT' = AC = b$

Instead of drawing perpendiculars to the two bisectors of the vertical angle either from the ends of the two sides or from the mid point of the base, if perpendiculars be drawn to the sides from certain points in the two bisectors of the vertical angle, values will be obtained for half the sum and half the difference of the sides.

FIGURE 9.

From U U′ let the perpendiculars US U′S′ be drawn to AC, and UT U′T′ to AB.

(7)* $\quad AS = AT = CS' = BT' = \frac{1}{2}(AC + AB)$

For the right-angled triangles UAS UAT are congruent;

therefore $\quad AS = AT \quad US = UT.$

Hence the right-angled triangles UCS UBT are congruent,

and $\quad CS = BT.$

* Parts of (7) and (8) are found in Leybourn's *Mathematical Repository*, old series, I. 283-4 (1799).

Similarly $AS' = AT' \quad CS' = BT'$.

Now $\tfrac{1}{2}(AC + AB) = \tfrac{1}{2}\{(AS + CS) + (AT - BT)\}$

$= \tfrac{1}{2}(AS + AT) = AS = AT$;

and also $= \tfrac{1}{2}\{(CS' + AS') + (BT' - AT')\}$

$= \tfrac{1}{2}(CS' + BT') = CS' = BT'$.

(8) $CS = BT = AS' = AT' = \tfrac{1}{2}(AC - AB)$

For $\tfrac{1}{2}(AC - AB) = \tfrac{1}{2}\{(AS + CS) - (AT - BT)\}$

$= \tfrac{1}{2}(CS + BT) = CS = BT$;

and also $= \tfrac{1}{2}\{(CS' + AS') - (BT' - AT')\}$

$= \tfrac{1}{2}(AS' + AT') = AS' = AT'$.

(9)
$$\begin{aligned} \angle U'BC &= U'CB = U'UB = U'UC \\ &= U'AS' = U'AT' = AUS = AUT \\ &= \tfrac{1}{2}(B + C) \end{aligned}$$

(10)
$$\begin{aligned} \angle U'BA &= U'CA = U'UA = BUT = CUS \\ &= \tfrac{1}{2}(B - C) \end{aligned}$$

For half the sum of two magnitudes increased by half their difference gives the greater.

(11) If $BP \; CQ$ be drawn perpendicular to AU,

$$\angle ABP = 90^\circ - \tfrac{1}{2}A = \tfrac{1}{2}(B + C) .$$

But $\angle AUS = \tfrac{1}{2}(B + C)$;

therefore $BP \; US$ intersect* on the circle ABC .

Similarly $CQ \; UT$ „ „ „ „ „ .

A like statement holds good for $BP' \; U'S'$, and for $CQ' \; U'T'$.

* Leybourn's *Mathematical Repository*, old series, I. 285 (1799).

(12)* If

$$\left.\begin{matrix} BP & US \\ CQ & UT \\ BP' & U'S' \\ CQ' & U'T' \end{matrix}\right\} \text{meet on the circumcircle at} \left\{\begin{matrix} B_2 \\ C_2 \\ B_2' \\ C_2' \end{matrix}\right.$$

$$\begin{aligned} 4US \cdot SB_2 &= 4UT \cdot TC_2 \\ &= 4U'S' \cdot S'B_2' = 4U'T' \cdot T'C_2' = AC^2 - AB^2 \end{aligned}$$

For
$$\begin{aligned} US \cdot SB_2 &= AS \cdot SC \\ &= \tfrac{1}{2}(AC + AB) \cdot \tfrac{1}{2}(AC - AB) \end{aligned}$$

(13) The incentre I of ABC is situated on AU. If with centre U and radius UI a circle be described, it will pass through B and C and will cut AC AB again at B_1 C_1 such that $B_1S = CS \quad C_1T = BT$.

Hence $B_1C = BC_1 = AC - AB$;

and $B\ P\ B_1$ are collinear, and so are $C\ Q\ C_1$.

(14)
$$\begin{aligned} U'B_2 &= UB_2' = AB_1 = AB \\ U'C_2 &= UC_2' = AC_1 = AC \end{aligned}$$

(15) B_2 B_2' are symmetrically situated with respect to O, and so are C_2 C_2'

(16) By reference to § 1, (12) it will be seen that

$$\angle AEP = AQE_1;$$

hence $E\ E_1\ P\ Q$ are concyclic.

Similarly $F\ F_1\ P\ Q$ „ „

The diameters of these two circles are EE_1 FF_1, and their centres are S T.

(17) In like manner $E_2\ E_3\ P'\ Q'$ are concyclic,

and also $F_2\ F_3\ P'\ Q'$ „ „

The diameters of these two circles are E_2E_3 F_2F_3, and their centres are S' T'.

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* Leybourn's *Mathematical Repository*, old series, I. 368 (1799).

(18) All the four circles are equal to each other, and their diameters are equal to BC.

The first two cut the circles I I_1 orthogonally,
and the second „ „ „ „ I_2 I_3 „ .

Compare § 1, (16).

§ 3.

To find values for the rectangles contained by various segments of the base BC.

Figure 9.

The values of the segments here given will be found useful in the verification of properties (1)–(12).

$$BX = \frac{a^2 - b^2 + c^2}{2a} \quad CX = \frac{a^2 + b^2 - c^2}{2a} \quad A'X = \frac{b^2 - c^2}{2a}$$

$$A'D = A'D_1 = \frac{b-c}{2}$$

$$A'D_2 = A'D_3 = \frac{b+c}{2}$$

$$BL = \frac{ca}{b+c} \qquad BL' = \frac{ca}{b-c}$$

$$CL = \frac{ab}{b+c} \qquad CL' = \frac{ab}{b-c}$$

$$A'L = \frac{a(b-c)}{2(b+c)} \qquad A'L' = \frac{a(b+c)}{2(b-c)}$$

$$LD = \frac{s_1(b-c)}{b+c} \qquad LD_1 = \frac{s(b-c)}{b+c}$$

$$L'D_2 = \frac{s_3(b+c)}{b-c} \qquad L'D_3 = \frac{s_2(b+c)}{b-c}$$

$$DX = \frac{s_1(b-c)}{a} \qquad D_1X = \frac{s(b-c)}{a}$$

$$D_2X = \frac{s_3(b+c)}{a} \qquad D_3X = \frac{s_2(b+c)}{a}$$

$$LX = \frac{2ss_1(b-c)}{a(b+c)} \qquad L'X = \frac{2s_2s_3(b+c)}{a(b-c)}$$

(1) $$A'X \cdot A'L = A'D^2 = A'D_1{}^2 = \tfrac{1}{4}(b-c)^2$$

Because A C Q X are concyclic

therefore $\angle A'XQ = \tfrac{1}{2}A = \angle A'QL$;

therefore triangles A'XQ A'QL are similar;

therefore $A'X : A'Q = A'Q : A'L$;

therefore
$$A'X \cdot A'L = A'Q^2$$
$$= A'D^2$$

(2) $$A'X \cdot A'L' = A'D_2{}^2 = A'D_3{}^2 = \tfrac{1}{4}(b+c)^2$$

This follows, in a manner analogous to the preceding, from the similarity of triangles A'XQ' A'Q'L'.

The following method may be used for proving (1) and (2).

Since the points A I L I_1 form a harmonic range, therefore their projections on BC will form a harmonic range; that is, X D L D_1 is a harmonic range.
Hence, since DD_1 is bisected at A',
$$A'X \cdot A'L = A'D^2 = A'D_1{}^2.$$

Similarly, since I_2 A I_3 L' form a harmonic range, so also will D_2 X D_3 L'.
Hence, since D_2D_3 is bisected at A',
$$A'X \cdot A'L' = A'D_2{}^2 = A'D_3{}^2.$$

(3) $$A'X \cdot LX = DX \cdot D_1X = \frac{ss_1(b-c)^2}{a^2}$$

For
$$\begin{aligned} A'X \cdot LX &= A'X^2 - A'X \cdot A'L \\ &= A'X^2 - A'D^2 \\ &= DX \cdot D_1X \end{aligned}$$

(4) $$A'X \cdot L'X = D_2X \cdot D_3X = \frac{s_2s_3(b+c)^2}{a^2}$$

For
$$\begin{aligned} A'X \cdot L'X &= A'X \,.\, A'L' - A'X^2 \\ &= A'D_3{}^2 - A'X^2 \\ &= D_2X \cdot D_3X \end{aligned}$$

$$(5) \qquad A'X \cdot LD = A'D \cdot DX = \frac{s_1(b-c)^2}{2a}$$

For
$$\begin{aligned} A'X \cdot LD &= A'X \cdot A'D - A'X \cdot A'L \\ &= A'X \cdot A'D - A'D^2 \\ &= A'D \cdot DX \end{aligned}$$

$$(6) \qquad A'X \cdot LD_1 = A'D_1 \cdot D_1X = \frac{s\,(b-c)^2}{2a}$$

For
$$\begin{aligned} A'X \cdot LD_1 &= A'X \cdot A'D_1 + A'X \cdot A'L \\ &= A'X \cdot A'D_1 + A'D_1{}^2 \\ &= A'D_1 \cdot D_1X \end{aligned}$$

$$(7) \qquad A'X \cdot L'D_3 = A'D_2 \cdot D_2X = \frac{s_3(b+c)^2}{2a}$$

For
$$\begin{aligned} A'X \cdot L'D_2 &= A'X \cdot A'L' + A'X \cdot A'D_2 \\ &= A'D_2{}^2 + A'X \cdot A'D_2 \\ &= A'D_2 \cdot D_2X \end{aligned}$$

$$(8) \qquad A'X \cdot L'D_3 = A'D_3 \cdot D_3X = \frac{s_2(b+c)^2}{2a}$$

For
$$\begin{aligned} A X \cdot L'D_3 &= A' X \cdot A'L' - A'X \cdot A'D_3 \\ &= A'D_3{}^2 - A'X \cdot A'D_3 \\ &= A'D_3 \cdot D_3X \end{aligned}$$

$$(9) \qquad A'L \;\; LX = DL \cdot LD_1 = \frac{ss_1(b-c)^2}{(b+c)^2}$$

For
$$\begin{aligned} A'L \;\; LX &= A'L \cdot A'X - A'L^2 \\ &= A'D^2 - A'L^2 \\ &= DL \cdot LD_1 \end{aligned}$$

$$(10) \qquad A'L' \cdot L'X = D_2L' \cdot L'D_3 = \frac{s_2s_3(b+c)^2}{(b-c)^2}$$

For
$$\begin{aligned} A'L' \cdot L'X &= A'L'^2 - A'L' \cdot A'X \\ &= A'L'^2 - A'D_3{}^2 \\ &= D_2L' \cdot L'D_3 \end{aligned}$$

(11) $$DX \cdot D_1X = BD \cdot DC - BX \cdot XC$$

For $$BD \cdot DC - BX \cdot XC = (A'B^2 - A'D^2) - (A'B^2 - A'X^2)$$
$$= A'X^2 - A'D^2$$
$$= DX \cdot D_1X$$

(12) $$D_2X \cdot D_3X = BD_2 \cdot D_2C + BX \cdot XC$$

For $$BD_2 \cdot D_2C - BX \cdot XC = (A'D_2^2 - A'C^2) + (A'C^2 - A'X^2)$$
$$= A'D_2^2 - A'X^2$$
$$= D_2X \cdot D_3X$$

In *Mathematical Questions from the Educational Times*, XIII. 34 (1870), T. T. Wilkinson says regarding (1):

"This is one of the properties of Halley's diagram, which was partially discussed in the four numbers of the *Student*, published at Liverpool from 1797 to 1800. It there forms Prop. 8, and is due to *Non Sibi*, a name assumed by the first editor, Mr John Knowles. In the diagram as there considered, the properties of one side only are given; but when all the sides are considered, there seems to be no limit to the relations between the different parts of the figure. Some time ago I considered the 'angular properties' only; and after writing down about 130 of them, they seemed to arise more abundantly than ever."

Halley's diagram somewhat resembles Figure 15, and it obtained that name, among the non-academic geometers of England, from the statement of W. Jones in his *Synopsis Palmariorum Matheseos*, p. 245 (1706), that he received it "from the learned Mr Halley." Jones says that an endless variety of useful theorems may be deduced from it, and that by inspection only.

The property (1), however, is older than the *Student*; for it is spoken of as a well-known theorem in the *Ladies' Diary* for 1785.

(3) and (7) occur in M'Dowell's *Exercises on Euclid*, §154 (1863); (9) and (11) are found in Leybourn's *Mathematical Repository*, old series, I. 369 (1799).

§ 4.

To find values for the rectangles contained by various segments of the diameter UU′.

FIGURE 15.

(1) $$A'U \cdot UK' = A'X \cdot A'L = \tfrac{1}{4}(b-c)$$

From the similar triangles UA′L AKU′

$$A'L : A'U = KU' : KA$$

that is, $$A'L : A'U = UK' : A'X$$

(2) $$A'U' \cdot U'K' = A'X \cdot A'L' = \tfrac{1}{4}(b+c)^2$$

From the similar triangles UA′L′ AKU

$$A'L' : A'U' = KU : KA$$

that is, $$A'L' : A'U' = U'K' : A'X$$

(3) $$A'K \cdot KU' = A'X \cdot LX$$

For
$$\begin{aligned} A'X \cdot LX &= DX \cdot D_1X \\ &= A'X^2 - A'D^2 \\ &= KA^2 - A'D^2 \\ &= UK \cdot KU' - A'U \cdot KU' \\ &= A'K \cdot KU' \end{aligned}$$

(4) $$A'K \cdot KU = A'X \cdot L'X$$

For
$$\begin{aligned} A'X \cdot L'X &= D_2X \cdot D_3X \\ &= A'D_2^2 - A'X^2 \\ &= A'D_2^2 - KA^2 \\ &= A'U' \cdot U'K' - UK \cdot KU' \\ &= A'K \cdot KU \end{aligned}$$

(5) $$A'K \cdot A'U = BD \cdot DC$$

For
$$\begin{aligned} BD \cdot DC &= A'C^2 - A'D^2 \\ &= A'U \cdot A'U' - A'U \cdot UK' \\ &= A'K \cdot A'U \end{aligned}$$

(6) $$A'K \cdot A'U' = BD_2 \cdot D_2C$$

For
$$\begin{aligned} BD_2 \cdot D_2C &= A'D_2{}^2 - A'C^2 \\ &= A'U' \cdot U'K' - A'U \cdot A'U' \\ &= A'K \cdot A'U' \end{aligned}$$

(7) $$A'K \cdot A'K' = BX \cdot XC$$

For
$$\begin{aligned} BX \cdot XC &= AX \cdot XR \\ &= A'K \cdot A'K' \end{aligned}$$

(8)* $$A'U \cdot UK = US^2$$

For
$$\begin{aligned} US^2 &= CU^2 - CS^2 \\ &= A'U \cdot UU' - A'U \cdot UK' \\ &= A'U \cdot U'K' \\ &= A'U \cdot UK \end{aligned}$$

(9) $$U'K \cdot ID = A'D \cdot DX$$

Since ID IL are respectively perpendicular to AK AU′, therefore the right-angled triangles AKU′ IDL are similar;

therefore $$U'K : AK = LD : ID$$

therefore
$$\begin{aligned} U'K \cdot ID &= AK \cdot LD \\ &= A'X \cdot LD \\ &= A'D \cdot DX \end{aligned}$$

(10)† $$U'K \cdot I_1D_1 = A'D_1 \cdot D_1X$$

The proof of this is similar to the preceding.

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* For (1), (2), (3), (5), (8) see Leybourn's *Mathematical Repository*, old series, I. 285, 368, 367, 369, 368 (1799).

† (9) and (10) are given by T. T. Wilkinson in *Mathematical Questions from the Educational Times*, XXIV. 28 (1875).

(11) $$ABC : XPQ = UU' : U'K$$
and $$ABC : XP'Q' = UU' : UK$$

FIGURE 9.

Draw $A'X'$ perpendicular to PQ;
then X' is the mid point of PQ.

Because $$\angle UCA' = \tfrac{1}{2}A = \angle A'PX',$$
therefore the right-angled triangles $UA'C$ $A'X'P$ are similar;

therefore $$A'C^2 : X'P^2 = UC^2 : A'P^2.$$

But $$\begin{aligned} ABC : XPQ &= BC^2 : PQ^2 \\ &= A'C^2 : X'P^2 ; \end{aligned}$$

therefore $$\begin{aligned} ABC : XPQ &= UC^2 : A'P^2 \\ &= A'U \cdot UU' : A'U \cdot UK' \\ &= UU' : UK' \\ &= UU' : U'K \end{aligned}$$

For another proof see § 1, (28).

In a similar manner it may be shown that

$$ABC : XP'Q' = UU' : UK.$$

(12) Because $$U'K + UK = UU'$$
another proof is obtained of the theorem that

$$ABC = XPQ + XP'Q'$$

(13) If the base BC and the vertical angle A be given, and if in AU AU' the bisectors of the interior and exterior angles at A, there be taken AP equal to half the sum, and AQ equal to half the difference of the sides, the loci of P and Q are two circles. If their radii be denoted by r' r'' and the radius of the circle inscribed in CUU' by r''', then

$$R = r' + r'' + r'''$$

Mr G. Robinson, jun., Hexham, in the *Lady's and Gentleman's Diary* for 1862, p. 74. Two solutions will be found in the *Diary* for 1863, pp. 49-50.

§ 5.

If through A' a perpendicular is drawn to BC, then AD AD_1 AD_2 AD_3 will intersect this perpendicular at R_1 R R_3 R_2 such that*

$$A'R = r \quad A'R_1 = r_1 \quad A'R_2 = r_2 \quad A'R_3 = r_3$$

FIGURE 16.

Let DI produced meet AD_1 at D'.

Since the line joining the extremities of two parallel and similarly directed radii cf two circles passes through their external homothetic centre; and since A is the external homothetic centre of the circles I_1 and I, and I_1D_1 ID' are parallel; therefore ID' is a radius of the incircle I, and $DD' = 2r$.

Now since $A'D = A'D_1$, and $A'R$ is parallel to DD', therefore $A'R = \frac{1}{2} DD' = r$.

Similarly for the other equalities.

(1) Through B', the mid point of CA, a perpendicular to CA is drawn, and this perpendicular is intersected by

	BE	BE_1	BE_2	BE_3
in the points	S_2	S_3	S	S_1 ;

through C', the mid point of AB, a perpendicular to AB is drawn, and this perpendicular is intersected by

	CF	CF_1	CF_2	CF_3
in the points	T_3	T_2	T_1	T respectively; then

$$B'S = r \quad B'S_1 = r_1 \quad B'S_2 = r_2 \quad B'S_3 = r_3$$
$$C'T = r \quad C'T_1 = r_1 \quad C'T_2 = r_2 \quad C'T_3 = r_3.$$

* W. H. Levy in the *Lady's and Gentleman's Diary* for 1863, p. 77, and for 1864, pp. 54-5.

(2) The four triangles RST $R_1S_1T_1$ $R_2S_2T_2$ $R_3S_3T_3$ are inversely similar to ABC; they have O, the circumcentre of ABC, for their common centre of homology, and OI OI_1 OI_2 OI_3 for the diameters of their circumcircles.

FIGURE 17.

Since $$A'R_1 = r_1 = I_1D_1$$
therefore $\angle I_1R_1A'$ is right.
Similarly $\angle I_1S_1B'$ and $\angle I_1T_1C'$ are right;
therefore the circle whose diameter is OI_1
passes through R_1 S_1 T_1.

Since R_1 S_1 O_1 T_1 are concyclic,

therefore
$$\begin{aligned}\angle S_1R_1T_1 &= 180^\circ - \angle S_1OT_1\\ &= 180 - \angle B'OC'\\ &= A;\end{aligned}$$

and
$$\angle R_1S_1T_1 = \angle R_1OT_1 = B,$$

since T_1O R_1O are respectively perpendicular to AB BC;
therefore triangle $R_1S_1T_1$ is similar to ABC.

(3) Let the mid points of OI OI_1 OI_2 OI_3
be denoted by I' I_1' I_2' I_3'
then $I_1'I_2'I_3'I'$ is an orthic tetrastigm, similar and similarly situated to the tetrastigm $I_1I_2I_3I$, and the radius of the circumcircle of any of its four triangles is R.

For the radius of the circumcircle of any of the four triangles of the orthic tetrastigm $I_1I_2I_3I$ is $2R$.

(4) Let AI BI CI meet the circumcircle of ABC in U V W, and let the points diametrically opposite to U V W be U' V' W'.

Then I is the orthocentre of the triangle UVW. Now since O is the circumcentre of UVW, therefore I' is the nine-point centre of the four triangles of the orthic tetrastigm $UVWI$.

In like manner since I_1 is the orthocentre, and O the circumcentre of the triangle $UV'W'$, I_1' is the nine-point centre of the four triangles of the orthic tetrastigm $UV'W'I_1$; and similarly for I_2' I_3'.

See *Proceedings of the Edinburgh Mathematical Society*, Vol. I., pp. 54–5 (1894).

(5) The sum of the circumcircles of the four RST triangles is three times the circumcircle of ABC.

Since circles are proportional to the squares of their diameters, the circumcircle of ABC is to the sum of the four RST circles as $4R^2$ is to $OI^2 + OI_1^2 + OI_2^2 + OI_3^2$.

Now
$$\begin{aligned}\Sigma(OI^2) &= 4R^2 + 2R(r_1 + r_2 + r_3 - r)\\ &= 4R^2 + 2R \cdot 4R\\ &= 12R^2.\end{aligned}$$

§ 6.

If UD UD_1 $U'D_2$ $U'D_3$ intersect AX at X_0 X_1 X_2 X_3, then*
$$XX_0 = r \quad XX_1 = r_1 \quad XX_2 = r_2 \quad XX_3 = r_3$$

FIGURE 18.

Through I draw a parallel to BC meeting UU′ in K_0 and AX in X_0; join UD DX_0.

Because
$$A'D^2 = A'X \cdot A'L$$
therefore
$$A'D : A'X = A'L : A'D$$
that is
$$\begin{aligned}A'D : K_0X_0 &= A'L : K_0I\\ &= UA' : UK_0\end{aligned}$$
therefore the points U D X_0 are collinear.

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* The first of these properties is given by W. Dixon Rangeley in the *Gentleman's Diary* for 1822, p. 47; the first and second (without any hint as to the third and fourth) by W. H. Levy in the *Lady's and Gentleman's Diary* for 1849, p. 75.

Similarly, if through I_1 a parallel be drawn to BC meeting AX in X_1, it may be proved that $U\ D_1\ X_1$ are collinear.

Through I_3 draw a parallel to BC meeting UU′ in K_3 and AX in X_3; join $U'D_3\ D_3X_3$.

Because $$A'D_3^2 = A'X \cdot A'L'$$
therefore $$A'D_3 : A'X = A'L' : A'D_3$$
that is $$A'D_3 : K_3X_3 = A'L' : K_3I_3$$
$$= U'A' : U'K_3\ ;$$
therefore the points $U'\ D_3\ X_3$ are collinear.

Similarly for the points $U'\ D_2\ X_2$.

(1) $VE\ \ V'E_1\ \ VE_2\ \ V'E_3$ intersect BY at $Y_0\ \ Y_1\ \ Y_2\ \ Y_3$; and

$WF\ \ W'F_1\ \ W'E_2\ \ WF_3$ intersect CZ at $Z_0\ \ Z_1\ \ Z_2\ \ Z_3$ such that

$$YY_0 = r \quad YY_1 = r_1 \quad YY_2 = r_2 \quad YY_3 = r_3$$
$$ZZ_0 = r \quad ZZ_1 = r_1 \quad ZZ_2 = r_2 \quad ZZ_3 = r_3.$$

(2) The four triangles $X_0Y_0Z_0\ \ X_1Y_1Z_1\ \ X_2Y_2Z_2\ \ X_3Y_3Z_3$ are inversely similar to ABC; they have H, the orthocentre of ABC, for their common centre of homology, and $HI\ HI_1\ HI_2\ HI_3$ for the diameters of their circumcircles.

FIGURE 19.

Since $$XX_1 = r_1 = I_1D_1$$
therefore $\angle\, I_1X_1X$ is right.
Similarly $\angle\, I_1Y_1Y$ and $\angle\, I_1Z_1Z$ are right;
therefore the circle whose diameter is HI
passes through $X_1\ Y_1\ Z_1$.

Since $X_1\ Y_1\ H\ Z_1$ are concyclic

therefore $$\angle Y_1X_1Z_1 = 180^\circ - \angle Y_1HZ_1$$
$$= A\ ;$$

and $$\angle X_1Y_1Z_1 = \angle X_1HZ_1 = B,$$

since $Z_1H\ X_1H$ are respectively perpendicular to AB BC ;
therefore triangle $X_1Y_1Z_1$ is similar to ABC.

(3) The mid points of $HI\ HI_1\ HI_2\ HI_3$ form an orthic tetrastigm similar and similarly situated to the tetrastigm $I_1I_2I_3I$, and the radius of the circumcircle of any of its four triangles is R.

(4) The sum of the circumcircles of the four triangles $X_0Y_0Z_0 \cdots\cdot$ is four times the sum of the circumcircles of the three triangles

AYZ XBZ XYC.

It will be seen from a subsequent Section that the values * of $HI^2 \cdots\cdot$ may be written

$$HI^2 = 4(R^2 - 2Rr\ ) + bc + ca + ab - (a^2 + b^2 + c^2)$$
$$HI_1^2 = 4(R^2 + 2Rr_1) + bc - ca - ab - (a^2 + b^2 + c^2)$$
$$HI_2^2 = 4(R^2 + 2Rr_2) - bc + ca - ab - (a^2 + b^2 + c^2)$$
$$HI_3^2 = 4(R^2 + 2Rr_3) - bc - ca + ab - (a^2 + b^2 + c^2)$$

Hence $$\Sigma(HI^2) = 4(12R^2 - a^2 - b^2 - c^2)$$
$$= 4(HA^2 + HB^2 + HC^2)$$

See *Proceedings of the Edinburgh Mathematical Society*, Vol. I., p. 63 (1894).

The statement that

$$HA^2 + HB^2 + HC^2 = 12R^2 - a^2 - b^2 - c^2$$

may be proved as follows.

FIGURE 20.

* The value of HI^2 is given by William Mawson in the *Lady's and Gentleman's Diary* for 1843, p. 75 ; the other values are given by William Rutherford and Samuel Bills in the *Diary* for 1844, p. 52.

The triangles CBZ AHZ are similar;

therefore $BC^2 : HA^2 = CZ^2 : AZ^2$;

therefore $BC^2 + HA^2 : BC^2 = CZ^2 + AZ^2 : CZ^2$

$= CA^2 : CZ^2$;

therefore $BC^2 + HA^2 = \dfrac{BC^2 \cdot CA^2}{CZ^2}$

$= 4R^2$

by a theorem of Brahmegupta.

Similarly $CA^2 + HB^2 = AB^2 + HC^2 = 4R^2$

For another proof see Feuerbach, *Eigenschaften...des...Dreiecks*, Section VI., Theorem 2.

§ 7.

If $A'I$ $A'I_1$ $A'I_2$ $A'I_3$ intersect AX at

X_0 X_1 X_2 X_3 then*

$$AX_0 = r \quad AX_1 = r_1 \quad AX_2 = r_2 \quad AX_3 = r_3$$

FIGURE 21.

Join CU, and draw the radius of the incircle IE.

Then $\angle UCA' = \angle IAE$;

therefore triangles CUA' AIE are similar;

therefore $CU : UA' = AI : IE$.

Now $CU : UA' = IU : UA'$

$= AI : AX_0$;

therefore $AX_0 = IE = r$

Similarly $AX_1 = r_1$.

* The first of these properties occurs incidentally in William Walker's proof of a theorem in the *Gentleman's Mathematical Companion* for 1803, p. 50.

If CU' be joined, and I_3E_3 the radius of the third excircle be drawn, then triangles $CU'A'$ AI_3E_3 will be similar, and since

$$CU' = I_3U',$$

it may be shown that $AX_3 = I_3E_3 = r_3$.

Corresponding to the four X points situated on AX, there will be four Y points, $Y_0\ Y_1\ Y_2\ Y_3$, situated on BY, and four Z points, $Z_0\ Z_1\ Z_2\ Z_3$ situated on CZ.

Some of the properties of this collection of points will be found in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I., pp. 89–96 (1894).

§ 8.

FIGURE 23.

If the medians AA' BB' CC' be intersected by

the radii			at the points		
DI	EI	FI	L_0	M_0	N_0
D_1I_1	E_1I_1	F_1I_1	L_1	M_1	N_1
D_2I_2	E_2I_2	F_2I_2	L_2	M_2	N_2
D_3I_3	E_3I_3	F_3I_3	L_3	M_3	N_3

then*

$$DL_0 = \frac{2\Delta}{b+c} \qquad EM_0 = \frac{2\Delta}{c+a} \qquad FN_0 = \frac{2\Delta}{a+b}$$

$$D_1L_1 = \frac{2\Delta}{b+c} \qquad E_1M_1 = -\frac{2\Delta}{a-c} \qquad F_1N_1 = -\frac{2\Delta}{a-b}$$

$$D_2L_2 = \frac{2\Delta}{b-c} \qquad E_2M_2 = -\frac{2\Delta}{c+a} \qquad F_2N_2 = \frac{2\Delta}{a-b}$$

$$D_3L_3 = \frac{2\Delta}{b-c} \qquad E_3M_3 = \frac{2\Delta}{a-c} \qquad F_3N_3 = -\frac{2\Delta}{a+b}$$

the distances of the L points from BC being considered positive when L is on the same side of BC as A, and negative when

* The first three values are given by W. H. Levy in the *Lady's and Gentleman's Diary* for 1859, p. 51.

it is on the opposite side from A. A similar convention holds for the M and the N points.

FIGURE 22.

Let AI meet BC at L; draw LS LT perpendicular to AC AB, and AX perpendicular to BC.

Then $$A'L : A'D = A'D : A'X$$
$$= L_0D : AX.$$

Now $$A'L : A'D = A'D - A'L : A'X - A'D$$
$$= LD : DX$$
$$= LI : IA$$
$$= LB : BA$$
$$= LT : AX;$$

therefore $$L_0D = LT = LS$$

Again $$LS \cdot AC + LT \cdot AB = 2ABC$$

therefore $$L_0D(b+c) = 2\Delta$$

Similarly for the other equalities

(1) L_0 M_0 N_0 lie on EF FD DE; and similarly for L_1 M_1 N_1 .....

A proof of this will be found in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I., pp. 57–8 (1894).

(2) $$\frac{1}{DL_0} + \frac{1}{D_1L_1} + \frac{1}{D_2L_2} + \frac{1}{D_3L_3}$$
$$= \frac{1}{EM_0} + \frac{1}{E_1M_1} + \frac{1}{E_2M_2} + \frac{1}{E_3M_3}$$
$$= \frac{1}{FN_0} + \frac{1}{F_1N_1} + \frac{1}{F_2N_2} + \frac{1}{F_3N_3} = 0$$

(3)*

$$\frac{1}{DL_0}+\frac{1}{EM_0}+\frac{1}{FN_0}=\frac{2}{r}$$

$$\frac{1}{D_1L_1}+\frac{1}{E_1M_1}+\frac{1}{F_1N_1}=-\frac{2}{h_1}$$

$$\frac{1}{D_2L_2}+\frac{1}{E_2M_2}+\frac{1}{F_2N_2}=-\frac{2}{h_2}$$

$$\frac{1}{D_3L_3}+\frac{1}{E_3M_3}+\frac{1}{F_3N_3}=-\frac{2}{h_3}$$

(4) The diagonals of the following pairs of parallelograms

$DL_0D_1L_1$	$D_2L_2D_3L_3$	intersect	at	A′
$EM_0E_2M_2$	$E_3M_3E_1M_1$	,,	,,	B′
$FN_0F_3N_3$	$F_1N_1F_2N_2$	,,	,,	C′

(5) The four LMN triangles are homologous, and their centre of homology is G the centroid of ABC.

(6)† $A'U : A'U - IL_0 = A'U : I_1L_1 - A'U = b+c : a$

$A'U' : I_2L_2 + A'U' = A'U' : I_3L_3 - A'U' = b-c : a$

FIGURE 23.

From I U draw IE US perpendicular to AC.

Then $AS = \frac{1}{2}(b+c) \quad AE = \frac{1}{2}(b+c-a)$.

Now

$$\begin{aligned} A'U : IL_0 &= UA : IA \\ &= AS : AE \\ &= b+c : b+c-a\,; \end{aligned}$$

therefore $A'U : A'U - IL_0 = b+c : a$

(7) $2A'U = IL_0 + I_1L_1$

$2A'U' = I_2L_2 - I_3L_3$

therefore $4R = IL_0 + I_1L_1 + I_2L_2 - I_3L_3$

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* The first result in (3) is given by W. H. Levy in the *Lady's and Gentleman's Diary* for 1858, p. 71.

† Of the four proportions in (6) the first is given by John Ryley, Leeds, in the *Gentleman's Mathematical Companion*, for 1802, p. 59. The solution in the text is that of J. H. Swale, Liverpool.

(8) If through $I \ I_1 \ I_2 \ I_3$ parallels be drawn to BC, meeting UU' in $K \ K_1 \ K_2 \ K_3$, then*

$$UK = UK_1 = US$$
$$U'K_2 = U'K_3 = U'S'$$

where $US \ U'S'$ are perpendicular to AC.

For the right-angled triangles CUS IUK are congruent,

since $$UC = UI,$$

and $$\angle CUS = \tfrac{1}{2}(B - C) = \angle IUK.$$

§ 9.

Formulae connected with the Angular Bisectors of a Triangle limited at their points of intersection with each other.

The notation

$$AI = \alpha \quad AI = \beta \quad CI = \gamma, \quad \text{etc.},$$

was suggested by T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 77, and adopted by Thomas Weddle in his admirable papers entitled "Symmetrical Properties of Plane Triangles," which appeared in the same publication (1843, 1845, 1848).

Neither Davies nor Weddle makes use of the equivalents for II_1, etc., namely $\alpha_1 - \alpha$, etc. Although the employment of these equivalents somewhat lengthens the formulae, it seems to me that it renders their symmetry a little more apparent.

In connection with the ascription, in the historical notes, of the great majority of the following formulae to Weddle, it is right to call attention to a letter of T. S. Davies in the *Lady's and Gentleman's Diary* for 1849, pp. 90-1, in which he states that when he undertook to arrange and systematise those properties of the triangle communicated to him, several sets of papers came into his hands, the most ample and elegant of which were those of Messrs Weddle and J. W. Elliott. The letter continues:

"I feel it to be due to him [Mr Elliott] to say that the names both of Mr Weddle and Mr Elliott might fairly have been prefixed to the far greater number of the properties, whilst each gentleman would have had a few properties designated as peculiar to himself."

I might have considerably shortened the lists of the formulae by giving only the leading identities, and referring the reader to Mr Lemoine's scheme of *continuous transformation*. I have done so here and there, but in general I have

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* The property that $UK = US$ is referred to as well known in the *Gentleman's Mathematical Companion* for 1803, p. 50.

either left or made the list complete in order that the reader who consults it may find ready to his hand the particular scrap of information of which he is in search. A table of logarithms which gave the logarithms of the prime numbers only would certainly be of some use, but merely to a select few.

Mr R. F. Muirhead suggests to me $t_1\ t_2\ t_3$ as a convenient mathematical translation of Mr Lemoine's *transformation continue en A, en B, en C.* Here is the t_1 applicable to the formulae which follow.

$$\begin{array}{cccccccccccccc} a & b & c & s & s_1 & s_2 & s_3 & r & r_1 & r_2 & r_3 & h_1 & h_2 & h_3 \end{array}$$

change into

$$\begin{array}{cccccccccccccc} a & -b & -c & -s_1 & -s & s_3 & s_2 & r_1 & r & -r_3 & -r_2 & -h_1 & h_2 & h_3 \end{array}$$

$$\begin{array}{ccccccccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{R} & \Delta & l_1 & l_2 & l_3 & \lambda_1 & \lambda_2 & \lambda_3 \end{array}$$

change into

$$\begin{array}{ccccccccccc} -\mathrm{A} & 180^\circ-\mathrm{B} & 180^\circ-\mathrm{C} & -\mathrm{R} & -\Delta & -l_1 & -\lambda_2 & -\lambda_3 & \lambda_1 & -l_2 & -l_3 \end{array}$$

$$\begin{array}{cccccccccccc} \alpha & \alpha_1 & \alpha_2 & \alpha_3 & \beta & \beta_1 & \beta_2 & \beta_3 & \gamma & \gamma_1 & \gamma_2 & \gamma_3 \end{array}$$

change into

$$\begin{array}{cccccccccccc} -\alpha_1 & -\alpha & -\alpha_3 & -\alpha_2 & \beta_1 & \beta & -\beta_3 & -\beta_2 & \gamma_1 & \gamma & -\gamma_3 & -\gamma_2 \end{array}$$

$$\left.\begin{array}{ll} \alpha^2 = (r_2-r)(r_3-r) & \alpha_1^2 = (r_3+r_1)(r_1+r_2) \\ \beta^2 = (r_3-r)(r_1-r) & \beta_1^2 = (r_1+r_2)(r_1-r) \\ \gamma^2 = (r_1-r)(r_2-r) & \gamma_1^2 = (r_1-r)(r_3+r_1) \\ & \\ \alpha_3^2 = (r_2-r)(r_1+r_2) & \alpha_3^2 = (r_3+r_1)(r_3-r) \\ \beta_2^2 = (r_1+r_2)(r_2+r_3) & \beta_3^2 = (r_3-r)(r_2+r_3) \\ \gamma_2^2 = (r_2+r_3)(r_2-r) & \gamma_3^2 = (r_2+r_3)(r_3+r_1) \end{array}\right\} \quad (1)$$

Weddle remarks that these twelve equations along with the three in (55) of the r formulae* give the values of all the products of every two of the six quantities

$$r_1-r \quad r_2-r \quad r_3-r \quad r_2+r_3 \quad r_3+r_1 \quad r_1+r_2$$

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* *Proceedings of the Edinburgh Mathematical Society,* Vol. XII., p. 98 (1894).

$$\left.\begin{array}{lll}
\alpha = \dfrac{\sqrt{bcrr_1}}{r_1} & \beta = \dfrac{\sqrt{carr_2}}{r_2} & \gamma = \dfrac{\sqrt{abrr_3}}{r_3} \\[2ex]
\alpha_1 = \dfrac{\sqrt{bcrr_1}}{r} & \beta_2 = \dfrac{\sqrt{carr_2}}{r} & \gamma_3 = \dfrac{\sqrt{abrr_3}}{r} \\[2ex]
\alpha_2 = \dfrac{\sqrt{bcr_2r_3}}{r_3} & \beta_3 = \dfrac{\sqrt{car_3r_1}}{r_1} & \gamma_1 = \dfrac{\sqrt{abr_1r_2}}{r_2} \\[2ex]
\alpha_3 = \dfrac{\sqrt{bcr_2r_3}}{r_2} & \beta_1 = \dfrac{\sqrt{car_3r_1}}{r_3} & \gamma_2 = \dfrac{\sqrt{abr_1r_2}}{r_1}
\end{array}\right\} \qquad (2)$$

$$\left.\begin{array}{ll}
\dfrac{\alpha}{r} = \dfrac{\alpha_1}{r_1} = \dfrac{\alpha_2}{s_3} = \dfrac{\alpha_3}{s_2} & \dfrac{\alpha}{s_1} = \dfrac{\alpha_1}{s} = \dfrac{\alpha_2}{r_2} = \dfrac{\alpha_3}{r_3} \\[2ex]
\dfrac{\beta}{r} = \dfrac{\beta_1}{s_3} = \dfrac{\beta_2}{r_2} = \dfrac{\beta_3}{s_1} & \dfrac{\beta}{s_2} = \dfrac{\beta_1}{r_1} = \dfrac{\beta_2}{s} = \dfrac{\beta_3}{r_3} \\[2ex]
\dfrac{\gamma}{r} = \dfrac{\gamma_1}{s_2} = \dfrac{\gamma_2}{s_1} = \dfrac{\gamma_3}{r_3} & \dfrac{\gamma}{s_3} = \dfrac{\gamma_1}{r_1} = \dfrac{\gamma_2}{r_2} = \dfrac{\gamma_3}{s}
\end{array}\right\} \qquad (3)$$

$$\alpha\alpha_1 = \alpha_2\alpha_3 = bc \quad \beta\beta_2 = \beta_3\beta_1 = ca \quad \gamma\gamma_3 = \gamma_1\gamma_2 = ab \qquad (4)$$

$$\alpha\alpha_1\alpha_2\alpha_3 = b^2c^2 \quad \beta\beta_1\beta_2\beta_3 = c^2a^2 \quad \gamma\gamma_1\gamma_2\gamma_3 = a^2b^2 \qquad (5)$$

$$\alpha\beta\gamma\alpha_1\beta_1\gamma_1\alpha_2\beta_2\gamma_2\alpha_3\beta_3\gamma_3 = (abc)^4 \qquad (6)$$

$$\left.\begin{array}{l}
\alpha\,\beta_1\gamma_3 = \alpha\,\beta_2\gamma_1 = \alpha_1\beta\,\gamma_2 = \alpha_1\beta_3\gamma \\
= \alpha_2\beta\,\gamma_3 = \alpha_2\beta_3\gamma_1 = \alpha_3\beta_1\gamma_2 = \alpha_3\beta_2\gamma = abc
\end{array}\right\} \qquad (7)$$

$$\left.\begin{array}{l}
\alpha\,\beta\,\gamma : abc = abc : \alpha_1\beta_2\gamma_3 \\
\alpha_1\beta_1\gamma_1 : abc = abc : \alpha\,\beta_3\gamma_2 \\
\alpha_2\beta_2\gamma_2 : abc = abc : \alpha_3\beta\,\gamma_1 \\
\alpha_3\beta_3\gamma_3 : abc = abc : \alpha_2\beta_1\gamma
\end{array}\right\} \qquad (8)$$

$$\left.\begin{aligned}
&\frac{\beta\,\gamma_2}{\alpha}=\frac{\beta_3\gamma}{\alpha}=\frac{\beta_1\gamma_3}{\alpha_1}=\frac{\beta_2\gamma_1}{\alpha_1}\\
&=\frac{\beta_1\gamma_2}{\alpha_2}=\frac{\beta_2\gamma}{\alpha_2}=\frac{\beta\,\gamma_3}{\alpha_3}=\frac{\beta_3\gamma_1}{\alpha_3}=a\\
&\frac{\alpha\,\gamma_1}{\beta}=\frac{\alpha_3\gamma}{\beta}=\frac{\alpha_1\gamma}{\beta_1}=\frac{\alpha_2\gamma_1}{\beta_1}\\
&=\frac{\alpha_1\gamma_2}{\beta_2}=\frac{\alpha_2\gamma_3}{\beta_2}=\frac{\alpha\,\gamma_3}{\beta_3}=\frac{\alpha_3\gamma_2}{\beta_3}=b\\
&\frac{\alpha\,\beta_1}{\gamma}=\frac{\alpha_2\beta}{\gamma}=\frac{\alpha_1\beta}{\gamma_1}=\frac{\alpha_3\beta_1}{\gamma_1}\\
&=\frac{\alpha\,\beta_2}{\gamma_2}=\frac{\alpha_2\beta_3}{\gamma_2}=\frac{\alpha_1\beta_3}{\gamma_3}=\frac{\alpha_3\beta_2}{\gamma_3}=c
\end{aligned}\right\}\quad(9)$$

$$\left.\begin{array}{ll}
\alpha\,:\beta\,=\gamma\,:r_1-r & \alpha_1:\beta_1=\gamma_1:r_1-r\\
\beta\,:\alpha\,=\gamma\,:r_2-r & \beta_1:\alpha_1=\gamma_1:r_3+r_1\\
\gamma\,:\alpha\,=\beta\,:r_3-r & \gamma_1:\alpha_1=\beta_1:r_1+r_2\\
\alpha_2:\beta_2=\gamma_2:r_2+r_3 & \alpha_3:\beta_3=\gamma_3:r_2+r_3\\
\beta_2:\alpha_2=\gamma_2:r_2-r & \beta_3:\alpha_3=\gamma_3:r_3+r_1\\
\gamma_2:\alpha_2=\beta_2:r_1+r_2 & \gamma_3:\alpha_3=\beta_3:r_3-r
\end{array}\right\}\quad(10)$$

$$\left.\begin{array}{ll}
\alpha_1:\beta_2=\gamma_3:r_2+r_3 & \alpha\,:\beta_3=\gamma_2:r_2+r_3\\
\beta_2:\alpha_1=\gamma_3:r_3+r_1 & \beta_3:\alpha\,=\gamma_2:r_2-r\\
\gamma_3:\alpha_1=\beta_2:r_1+r_2 & \gamma_2:\alpha\,=\beta_3:r_3-r\\
\alpha_3:\beta\,=\gamma_1:r_1-r & \alpha_2:\beta_1=\gamma\,:r_1-r\\
\beta\,:\alpha_3=\gamma_1:r_3+r_1 & \beta_1:\alpha_2=\gamma\,:r_2-r\\
\gamma_1:\alpha_3=\beta\,:r_3-r & \gamma\,:\alpha_2=\beta_1:r_1+r_2
\end{array}\right\}\quad(11)$$

$$\left.\begin{array}{ll}
\alpha^2\,:bc=s_1:s & \alpha_1^2:bc=s\,:s_1\\
\beta^2\,:ca=s_2:s & \beta_1^2:ca=s_3:s_1\\
\gamma^2\,:ab=s_3:s & \gamma_1^2:ab=s_2:s_1\\
\alpha_2^2:bc=s_3:s_2 & \alpha_3^2:bc=s_2:s_3\\
\beta_2^2:ca=s\,:s_2 & \beta_3^2:ca=s_1:s_3\\
\gamma_2^2:ab=s_1:s_2 & \gamma_3^2:ab=s\,:s_3
\end{array}\right\}\quad(12)$$

$$\left.\begin{aligned}
\frac{\alpha^2}{bc}+\frac{\beta^2}{ca}+\frac{\gamma^2}{ab}&=1\\
\frac{\alpha_1^{\,2}}{bc}-\frac{\beta_1^{\,2}}{ca}-\frac{\gamma_1^{\,2}}{ab}&=1\\
-\frac{\alpha_2^{\,2}}{bc}+\frac{\beta_2^{\,2}}{ca}-\frac{\gamma_2^{\,2}}{ab}&=1\\
-\frac{\alpha_3^{\,2}}{bc}-\frac{\beta_3^{\,2}}{ca}+\frac{\gamma_3^{\,2}}{ab}&=1
\end{aligned}\right\}\qquad(13)$$

$$\left.\begin{aligned}
\frac{bc}{\alpha_1^{\,2}}+\frac{ca}{\beta_2^{\,2}}+\frac{ab}{\gamma_3^{\,2}}&=1\\
\frac{bc}{\alpha^2}-\frac{ca}{\beta_3^{\,2}}-\frac{ab}{\gamma_2^{\,2}}&=1\\
-\frac{bc}{\alpha_3^{\,2}}+\frac{ca}{\beta^2}-\frac{ab}{\gamma_1^{\,2}}&=1\\
-\frac{bc}{\alpha_2^{\,2}}-\frac{ca}{\beta_1^{\,2}}+\frac{ab}{\gamma^2}&=1
\end{aligned}\right\}\qquad(14)$$

$$\left.\begin{aligned}
\alpha^2\left(\frac{1}{c}-\frac{1}{b}\right)+\beta^2\left(\frac{1}{a}-\frac{1}{c}\right)+\gamma^2\left(\frac{1}{b}-\frac{1}{a}\right)&=0\\
\alpha_1^{\,2}\left(\frac{1}{b}-\frac{1}{c}\right)+\beta_1^{\,2}\left(\frac{1}{a}+\frac{1}{c}\right)-\gamma_1^{\,2}\left(\frac{1}{a}+\frac{1}{b}\right)&=0
\end{aligned}\right\}\qquad(15$$

$$\left.\begin{aligned}
a\alpha^2(b-c)+b\beta^2(c-a)+c\gamma^2(a-b)&=0\\
a\alpha_1^{\,2}(c-b)+b\beta_1^{\,2}(c+a)+c\gamma_1^{\,2}(a+b)&=0
\end{aligned}\right\}\qquad(16)$$

$$\left.\begin{aligned}
\frac{b-c}{a\alpha_1^{\,2}}+\frac{c-a}{b\beta_2^{\,2}}+\frac{a-b}{c\gamma_3^{\,2}}&=0\\
\frac{c-b}{a\alpha^2}+\frac{c+a}{b\beta_3^{\,2}}-\frac{a+b}{c\gamma_2^{\,2}}&=0
\end{aligned}\right\}\qquad(17)$$

$$\left.\begin{aligned}
\alpha^2 + \beta^2 + \gamma^2 &= \frac{bcs_1 + cas_2 + abs_3}{s} = \quad bc + ca + ab - \frac{3abc}{s} \\
\alpha_1^2 + \beta_1^2 + \gamma_1^2 &= \frac{bcs + cas_3 + abs_2}{s_1} = \quad bc - ca - ab + \frac{3abc}{s_1} \\
\alpha_2^2 + \beta_2^2 + \gamma_2^2 &= \frac{bcs_3 + cas + abs_1}{s_2} = -bc + ca - ab + \frac{3abc}{s_2} \\
\alpha_3^2 + \beta_3^2 + \gamma_3^2 &= \frac{bcs_2 + cas_1 + abs}{s_3} = -bc - ca + ab + \frac{3abc}{s_3}
\end{aligned}\right\} \quad (18)$$

$$\left.\begin{aligned}
\alpha_1^2 + \beta_2^2 + \gamma_3^2 &= (r_1 + r_2 + r_3)^2 + s^2 \\
\alpha^2 + \beta_3^2 + \gamma_2^2 &= (r - r_3 - r_2)^2 + s_1^2 \\
\alpha_3^2 + \beta^2 + \gamma_1^2 &= (r - r_1 - r_3)^2 + s_2^2 \\
\alpha_2^2 + \beta_1^2 + \gamma^2 &= (r - r_2 - r_1)^2 + s_3^2
\end{aligned}\right\} \quad (19)$$

Compare (15) of the r formulae*

$$\left.\begin{aligned}
\alpha\beta\gamma &: abc = r : s \\
\alpha_1\beta_1\gamma_1 &: abc = r_1 : s_1 \\
\alpha_2\beta_2\gamma_2 &: abc = r_2 : s_2 \\
\alpha_3\beta_3\gamma_3 &: abc = r_3 : s_3
\end{aligned}\right\} \quad (20)$$

Other proportions may be obtained by substituting for abc its equivalents in (7). Matthes (p. 49) gives

$$\alpha_3\beta_1\gamma_2 : \alpha\beta\gamma = \Delta : r^2$$

which may be reduced to

$$abc : \alpha\beta\gamma = s : r$$

$$\left.\begin{aligned}
\alpha_1\beta_2\gamma_3 &: abc = s : r \\
\alpha\beta_3\gamma_2 &: abc = s_1 : r_1 \\
\alpha_3\beta\gamma_1 &: abc = s_2 : r_2 \\
\alpha_2\beta_1\gamma &: abc = s_3 : r_3
\end{aligned}\right\} \quad (21)$$

$$\left.\begin{aligned}
h_1h_2h_3\alpha\beta\gamma &= 8\Delta^2r^2 \qquad & h_1h_2h_3\alpha_1\beta_2\gamma_3 &= 8\Delta^2s^2 \\
h_1h_2h_3\alpha_1\beta_1\gamma_1 &= 8\Delta^2r_1^2 \qquad & h_1h_2h_3\alpha\beta_3\gamma_2 &= 8\Delta^2s_1^2
\end{aligned}\right\} \quad (22)$$

and so on.

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* *Proceedings of the Edinburgh Mathematical Society*, Vol. XII., p. 91 (1894).

$$\left.\begin{aligned} a\alpha^2 + b\beta^2 + c\gamma^2 &= \\ a\alpha_1^2 - b\beta_1^2 - c\gamma_1^2 &= \\ -a\alpha_2^2 + b\beta_2^2 - c\gamma_2^2 &= \\ -a\alpha_3^2 - b\beta_3^2 + c\gamma_3^2 &= abc \end{aligned}\right\} \quad (23)$$

$$\left.\begin{aligned} \alpha\,\beta\,\gamma &= (r_1 - r)(r_2 - r)(r_3 - r) \\ \alpha_1\beta_1\gamma_1 &= (r_1 - r)(r_3 + r_1)(r_1 + r_2) \\ \alpha_2\beta_2\gamma_2 &= (r_2 + r_3)(r_2 - r)(r_1 + r_2) \\ \alpha_3\beta_3\gamma_3 &= (r_2 + r_3)(r_3 + r_1)(r_3 - r) \\ \alpha_1\beta_2\gamma_3 &= (r_2 + r_3)(r_3 + r_1)(r_1 + r_2) \\ \alpha\,\beta_3\gamma_2 &= (r_2 + r_3)(r_2 - r)(r_3 - r) \\ \alpha_3\beta\,\gamma_1 &= (r_1 - r)(r_3 + r_1)(r_3 - r) \\ \alpha_2\beta_1\gamma &= (r_1 - r)(r_2 - r)(r_1 + r_2) \end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned} a\,\beta\,\beta_3 &= (r_2 + r_3)(r_3 - r)(r_1 - r) \\ a\,\gamma\,\gamma_2 &= (r_2 + r_3)(r_1 - r)(r_2 - r) \\ b\,\gamma\,\gamma_1 &= (r_3 + r_1)(r_1 - r)(r_2 - r) \\ b\,\alpha\,\alpha_3 &= (r_3 + r_1)(r_2 - r)(r_3 - r) \\ c\,\alpha\,\alpha_2 &= (r_1 + r_2)(r_2 - r)(r_3 - r) \\ c\,\beta\,\beta_1 &= (r_1 + r_2)(r_3 - r)(r_1 - r) \\ a\,\beta_1\beta_2 &= (r_1 - r)(r_1 + r_2)(r_2 + r_3) \\ a\,\gamma_3\gamma_1 &= (r_1 - r)(r_2 + r_3)(r_3 + r_1) \\ b\,\gamma_2\gamma_3 &= (r_2 - r)(r_2 + r_3)(r_3 + r_1) \\ b\,\alpha_1\alpha_2 &= (r_2 - r)(r_3 + r_1)(r_1 + r_2) \\ c\,\alpha_3\alpha_1 &= (r_3 - r)(r_3 + r_1)(r_1 + r_2) \\ c\,\beta_2\beta_3 &= (r_3 - r)(r_1 + r_2)(r_2 + r_3) \end{aligned}\right\} \quad (25)$$

Weddle remarks that (24) and (25) exhibit the twenty products of every three of the six quantities

$$r_1 - r, \quad r_2 - r, \quad r_3 - r, \quad r_2 + r_3, \quad r_3 + r_1, \quad r_1 + r_2$$

$$\left.\begin{array}{ll}
\alpha\,\alpha_2=(r_2-r)c & \alpha_3\,\alpha_1=(r_3+r_1)c \\
\alpha\,\alpha_3=(r_3-r)b & \alpha_1\,\alpha_2=(r_1+r_2)b \\
\beta\beta_3=(r_3-r)a & \beta_1\beta_2=(r_1+r_2)a \\
\beta\beta_1=(r_1-r)c & \beta_2\beta_3=(r_2+r_3)c \\
\gamma\gamma_1=(r_1-r)b & \gamma_2\gamma_3=(r_2+r_3)b \\
\gamma\gamma_2=(r_2-r)a & \gamma_3\gamma_1=(r_3+r_1)a
\end{array}\right\}\quad(26)$$

$$\left.\begin{array}{ll}
\alpha\,b=\alpha_3(r_2-r) & \alpha_2 b=\alpha_1(r_2-r) \\
\alpha\,c=\alpha_2(r_3-r) & \alpha_2 c=\alpha\,(r_1+r_2) \\
\beta\,c=\beta_1(r_3-r) & \beta_2 c=\beta_3(r_1+r_2) \\
\beta\,a=\beta_3(r_1-r) & \beta_2 a=\beta_1(r_2+r_3) \\
\gamma\,a=\gamma_2(r_1-r) & \gamma_2 a=\gamma\,(r_2+r_3) \\
\gamma\,b=\gamma_1(r_2-r) & \gamma_2\,b=\gamma_3(r_2-r) \\
\alpha_1 b=\alpha_2\,(r_3+r_1) & \alpha_3\,b=\alpha\,(r_3+r_1) \\
\alpha_1 c=\alpha_3\,(r_1+r_2) & \alpha_3\,c=\alpha_1\,(r_3-r) \\
\beta_1\,c=\beta\,(r_1+r_2) & \beta_3\,c=\beta_2(r_3-r) \\
\beta_1 a=\beta_2(r_1-r) & \beta_3 a=\beta\,(r_2+r_3) \\
\gamma_1 a=\gamma_3(r_1-r) & \gamma_3 a=\gamma_1(r_2+r_3) \\
\gamma_1\,b=\gamma\,(r_3+r_1) & \gamma_3 b=\gamma_2(r_3+r_1)
\end{array}\right\}\quad(27)$$

$$\left.\begin{array}{l}
(\alpha_1-\alpha)r=\alpha\,(r_1-r)=\beta\,\gamma \\
(\alpha_1-\alpha)r_1=\alpha_1(r_1-r)=\beta_1\gamma_1 \\
(\beta_2-\beta)r=\beta\,(r_2-r)=\gamma\,\alpha \\
(\beta_2-\beta)r_2=\beta_2(r_2-r)=\gamma_2\alpha_2 \\
(\gamma_3-\gamma)r=\gamma\,(r_3-r)=\alpha\,\beta \\
(\gamma_3-\gamma)r_3=\gamma_3(r_3-r)=\alpha_3\beta_3 \\
(\alpha_2+\alpha_3)r_2=\alpha_2(r_2+r_3)=\beta_2\gamma_2 \\
(\alpha_2+\alpha_3)r_3=\alpha_3(r_2+r_3)=\beta_3\gamma_3 \\
(\beta_3+\beta_1)r_3=\beta_3(r_3+r_1)=\gamma_3\alpha_3 \\
(\beta_3+\beta_1)r_1=\beta_1(r_3+r_1)=\gamma_1\alpha_1 \\
(\gamma_1+\gamma_2)r_1=\gamma_1(r_1+r_2)=\alpha_1\beta_1 \\
(\gamma_1+\gamma_2)r_2=\gamma_2(r_1+r_2)=\alpha_2\beta_2
\end{array}\right\}\quad(28)$$

$$\left.\begin{aligned}
(\alpha_1 - \alpha)s_2 &= \alpha_3(r_1 - r) = \beta\gamma_1 \\
(\alpha_1 - \alpha)s_3 &= \alpha_2(r_1 - r) = \beta_1\gamma \\
(\beta_2 - \beta)s_3 &= \beta_1(r_2 - r) = \gamma\alpha_2 \\
(\beta_2 - \beta)s_1 &= \beta_3(r_2 - r) = \gamma_2\alpha \\
(\gamma_3 - \gamma)s_1 &= \gamma_2(r_3 - r) = \alpha\beta_3 \\
(\gamma_3 - \gamma)s_2 &= \gamma_1(r_3 - r) = \alpha_3\beta \\
(\alpha_2 + \alpha_3)s &= \alpha_1(r_2 + r_3) = \beta_2\gamma_3 \\
(\alpha_2 + \alpha_3)s_1 &= \alpha(r_2 + r_3) = \beta_3\gamma_2 \\
(\beta_3 + \beta_1)s &= \beta_2(r_3 + r_1) = \gamma_3\alpha_1 \\
(\beta_3 + \beta_1)s_2 &= \beta(r_3 + r_1) = \gamma_1\alpha_3 \\
(\gamma_1 + \gamma_2)s &= \gamma_3(r_1 + r_2) = \alpha_1\beta_2 \\
(\gamma_1 + \gamma_2)s_3 &= \gamma(r_1 + r_2) = \alpha_2\beta_1
\end{aligned}\right\} \quad (29)$$

$$\left.\begin{aligned}
(\alpha_1 - \alpha)h_1 &= 2\alpha r_1 = 2\alpha_1 r = 2\alpha_2 s_2 = 2\alpha_3 s_3 \\
(\beta_2 - \beta)h_2 &= 2\beta r_2 = 2\beta_2 r = 2\beta_1 s_1 = 2\beta_3 s_3 \\
(\gamma_3 - \gamma)h_3 &= 2\gamma r_3 = 2\gamma_3 r = 2\gamma_1 s_1 = 2\gamma_2 s_2 \\
(\alpha_2 + \alpha_3)h_1 &= 2\alpha s = 2\alpha_1 s_1 = 2\alpha_2 r_3 = 2\alpha_3 r_2 \\
(\beta_3 + \beta_1)h_2 &= 2\beta s = 2\beta_2 s_2 = 2\beta_1 r_3 = 2\beta_3 r_1 \\
(\gamma_1 + \gamma_2)h_3 &= 2\gamma s = 2\gamma_3 s_3 = 2\gamma_1 r_2 = 2\gamma_2 r_1
\end{aligned}\right\} \quad (30)$$

$$\left.\begin{aligned}
(\alpha_1 - \alpha)a &= (\alpha_2 + \alpha_3)(r_1 - r) \\
(\beta_2 - \beta)b &= (\beta_3 + \beta_1)(r_2 - r) \\
(\gamma_3 - \gamma)c &= (\gamma_1 - \gamma_2)(r_3 - r) \\
(\alpha_2 + \alpha_3)a &= (\alpha_1 - \alpha)(r_2 + r_3) \\
(\beta_3 + \beta_1)b &= (\beta_2 - \beta)(r_3 + r_1) \\
(\gamma_1 + \gamma_2)c &= (\gamma_3 - \gamma)(r_1 + r_2)
\end{aligned}\right\} \quad (31)$$

$$
\left.\begin{aligned}
(\alpha_1 - \alpha)b &= (\beta_2 - \beta)\gamma_1 = (\beta_3 + \beta_1)\gamma \\
(\alpha_1 - \alpha)c &= (\gamma_3 - \gamma)\beta_1 = (\gamma_1 + \gamma_2)\beta \\
(\beta_2 - \beta)c &= (\gamma_3 - \gamma)\alpha_2 = (\gamma_1 + \gamma_2)\alpha \\
(\beta_2 - \beta)a &= (\alpha_1 - \alpha)\gamma_2 = (\alpha_2 + \alpha_3)\gamma \\
(\gamma_3 - \gamma)a &= (\alpha_1 - \alpha)\beta_3 = (\alpha_2 + \alpha_3)\beta \\
(\gamma_3 - \gamma)b &= (\beta_2 - \beta)\alpha_3 = (\beta_3 + \beta_1)\alpha \\
(\alpha_2 + \alpha_3)b &= (\beta_3 + \beta_1)\gamma_2 = (\beta_2 - \beta)\gamma_3 \\
(\alpha_2 + \alpha_3)c &= (\gamma_1 + \gamma_2)\beta_3 = (\gamma_3 - \gamma)\beta_2 \\
(\beta_3 + \beta_1)c &= (\gamma_1 + \gamma_2)\alpha_3 = (\gamma_3 - \gamma)\alpha_2 \\
(\beta_3 + \beta_1)a &= (\alpha_2 + \alpha_3)\gamma_1 = (\alpha_1 - \alpha)\gamma_3 \\
(\gamma_1 + \gamma_2)a &= (\alpha_2 + \alpha_3)\beta_1 = (\alpha_1 - \alpha)\beta_2 \\
(\gamma_1 + \gamma_2)b &= (\beta_3 + \beta_1)\alpha_2 = (\beta_2 - \beta)\alpha_1
\end{aligned}\right\} \tag{32}
$$

$$
\left.\begin{aligned}
(\alpha_1 - \alpha)\beta &= (\gamma_3 - \gamma)(r_1 - r) \\
(\alpha_1 - \alpha)\gamma &= (\beta_2 - \beta)(r_1 - r) \\
(\alpha_1 - \alpha)\beta_1 &= (\gamma_1 + \gamma_2)(r_1 - r) \\
(\alpha_1 - \alpha)\gamma_1 &= (\beta_3 + \beta_1)(r_1 - r) \\
(\beta_2 - \beta)\gamma &= (\alpha_1 - \alpha)(r_2 - r) \\
(\beta_2 - \beta)\alpha &= (\gamma_3 - \gamma)(r_2 - r) \\
(\beta_2 - \beta)\gamma_2 &= (\alpha_2 + \alpha_3)(r_2 - r) \\
(\beta_2 - \beta)\alpha_2 &= (\gamma_1 + \gamma_2)(r_2 - r) \\
(\gamma_3 - \gamma)\alpha &= (\beta_2 - \beta)(r_3 - r) \\
(\gamma_3 - \gamma)\beta &= (\alpha_1 - \alpha)(r_3 - r) \\
(\gamma_3 - \gamma)\alpha_3 &= (\beta_3 + \beta_1)(r_3 - r) \\
(\gamma_3 - \gamma)\beta_3 &= (\alpha_2 + \alpha_3)(r_3 - r)
\end{aligned}\right\} \tag{33}
$$

$$\left.\begin{aligned}
(\alpha_2+\alpha_3)\beta_2 &= (\gamma_1+\gamma_2)(r_2+r_3)\\
(\alpha_2+\alpha_3)\gamma_2 &= (\beta_2-\beta)(r_2+r_3)\\
(\alpha_2+\alpha_3)\beta_3 &= (\gamma_3-\gamma)(r_2+r_3)\\
(\alpha_2+\alpha_3)\gamma_3 &= (\beta_3+\beta_1)(r_2+r_3)\\[1ex]
(\beta_3+\beta_1)\gamma_1 &= (\alpha_1-\alpha)(r_3+r_1)\\
(\beta_3+\beta_1)\alpha_1 &= (\gamma_1+\gamma_2)(r_3+r_1)\\
(\beta_3+\beta_1)\gamma_3 &= (\alpha_2+\alpha_3)(r_3+r_1)\\
(\beta_3+\beta_1)\alpha_3 &= (\gamma_3-\gamma_1)(r_3+r_1)\\[1ex]
(\gamma_1+\gamma_2)\alpha_1 &= (\beta_3+\beta_1)(r_1+r_2)\\
(\gamma_1+\gamma_2)\beta_1 &= (\alpha_1-\alpha)(r_1+r_2)\\
(\gamma_1+\gamma_2)\alpha_2 &= (\beta_2-\beta)(r_1+r_2)\\
(\gamma_1+\gamma_2)\beta_2 &= (\alpha_2+\alpha_3)(r_1+r_2)
\end{aligned}\right\}\quad(34)$$

$$\left.\begin{aligned}
(\alpha_1-\alpha)r_2 &= (\alpha_2+\alpha_3)s_3 = a\alpha_2 = \beta_1\gamma_2 = \beta_2\gamma\\
(\alpha_1-\alpha)r_3 &= (\alpha_2+\alpha_3)s_2 = a\alpha_3 = \beta\gamma_3 = \beta_3\gamma_1\\
(\beta_2-\beta)r_3 &= (\beta_3+\beta_1)s_1 = b\beta_3 = \gamma_3\alpha = \gamma_2\alpha_3\\
(\beta_2-\beta)r_1 &= (\beta_3+\beta_1)s_3 = b\beta_1 = \gamma\alpha_1 = \gamma_1\alpha_2\\
(\gamma_3-\gamma)r_1 &= (\gamma_1+\gamma_2)s_2 = c\gamma_1 = \alpha_1\beta = \alpha_3\beta_1\\
(\gamma_3-\gamma)r_2 &= (\gamma_1+\gamma_2)s_1 = c\gamma_2 = \alpha\beta_2 = \alpha_2\beta_3\\[1ex]
(\alpha_2+\alpha_3)r &= (\alpha_1-\alpha)s_1 = a\alpha = \beta\gamma_2 = \beta_3\gamma\\
(\alpha_2+\alpha_3)r_1 &= (\alpha_1-\alpha)s = a\alpha_1 = \beta_1\gamma_3 = \beta_2\gamma_1\\
(\beta_3+\beta_1)r &= (\beta_2-\beta)s_2 = b\beta = \gamma_1\alpha = \gamma\alpha_3\\
(\beta_3+\beta_1)r_2 &= (\beta_2-\beta)s = b\beta_2 = \gamma_2\alpha_1 = \gamma_3\alpha_2\\
(\gamma_1+\gamma_2)r &= (\gamma_3-\gamma)s_3 = c\gamma = \alpha\beta_1 = \alpha_2\beta\\
(\gamma_1+\gamma_2)r_3 &= (\gamma_3-\gamma)s = c\gamma_3 = \alpha_1\beta_3 = \alpha_3\beta_2
\end{aligned}\right\}\quad(35)$$

$$\left.\begin{aligned}
(\alpha_1-\alpha)^2 &= a^2+(r_1-r)^2=\frac{a^2bcrr_1}{\Delta^2}=\frac{a^2bc}{ss_1}\\
&= (r_1-r)\frac{(r_2+r_3)(r_3+r_1)(r_1+r_2)}{r_2r_3+r_3r_1+r_1r_2}\\
(\beta_2-\beta)^2 &= b^2+(r_2-r)^2=\frac{ab^2crr_2}{\Delta^2}=\frac{ab^2c}{ss_2}\\
&= (r_2-r)\ [\qquad\qquad]\\
(\gamma_3-\gamma)^2 &= c^2+(r_3-r)^2=\frac{abc^2rr_3}{\Delta^2}=\frac{abc^2}{ss_3}\\
&= (r_3-r)\ [\qquad\qquad]\\
(\alpha_2+\alpha_3)^2 &= a^2+(r_2+r_3)^2=\frac{a^2bcr_2r_3}{\Delta^2}=\frac{a^2bc}{s_2s_3}\\
&= (r_2+r_3)\ [\qquad\qquad]\\
(\beta_3+\beta_1)^2 &= b^2+(r_3+r_1)^2=\frac{ab^2cr_3r_1}{\Delta^2}=\frac{ab^2c}{s_3s_1}\\
&= (r_3+r_1)\ [\qquad\qquad]\\
(\gamma_1+\gamma_2)^2 &= c^2+(r_1+r_2)^2=\frac{abc^2r_1r_2}{\Delta^2}=\frac{abc^2}{s_1s_2}\\
&= (r_1+r_2)\ [\qquad\qquad]
\end{aligned}\right\}\qquad(36)$$

In Grunert's *Archiv*, XXIX., 436 (1857), Franz Unferdinger gives for $(\alpha_1-\alpha)^2$, etc., the values

$$r_1^2(r_2+r_3)\frac{(r_2+r_3)(r_3+r_1)(r_1+r_2)}{(r_2r_3+r_3r_1+r_1r_2)^2}\ \text{etc.}\qquad(37)$$

See (56) of the r formulae.*

$$\left.\begin{aligned}
&(\alpha_1-\alpha)^2+(\beta_2-\beta)^2+(\gamma_3-\gamma)^2\\
&+(\alpha_2+\alpha_3)^2+(\beta_3+\beta_1)^2+(\gamma_1+\gamma_2)^2\\
&\qquad=3(-r+r_1+r_2+r_3)^2
\end{aligned}\right\}\qquad(38)$$

* *Proceedings of the Edinburgh Mathematical Society*, Vol. XII., p. 98 (1894).

$$\left.\begin{aligned}
(\alpha_1-\alpha)^2 &= (\beta_3+\beta_1)\beta_1-(\beta_2-\beta)\beta\\
&= (\gamma_1+\gamma_2)\gamma_2-(\gamma_3-\gamma)\gamma\\
(\beta_2-\beta)^2 &= (\gamma_1+\gamma_2)\gamma_2-(\gamma_3-\gamma)\gamma\\
&= (\alpha_2+\alpha_3)\alpha_2-(\alpha_1-\alpha)\alpha\\
(\gamma_3-\gamma)^2 &= (\alpha_2+\alpha_3)\alpha_3-(\alpha_1-\alpha)\alpha\\
&= (\beta_3+\beta_1)\beta_3-(\beta_2-\beta)\beta_1\\
(\alpha_2+\alpha_3)^2 &= (\beta_3+\beta_1)\beta_3+(\beta_2-\beta)\beta_2\\
&= (\gamma_1+\gamma_2)\gamma_2+(\gamma_3-\gamma)\gamma_3\\
(\beta_3+\beta_1)^2 &= (\gamma_1+\gamma_2)\gamma_1+(\gamma_3-\gamma)\gamma_3\\
&= (\alpha_2+\alpha_3)\alpha_3+(\alpha_1-\alpha)\alpha_1\\
(\gamma_1+\gamma_2)^2 &= (\alpha_2+\alpha_3)\alpha_2+(\alpha_1-\alpha)\alpha_1\\
&= (\beta_3+\beta_1)\beta_1+(\beta_2-\beta)\beta_2
\end{aligned}\right\}\quad(39)$$

$$\left.\begin{aligned}
(\alpha_1-\alpha)(\alpha_2+\alpha_3) &= (\beta_3+\beta_1)\beta_2-(\beta_2-\beta)\beta_3\\
&= (\gamma_1+\gamma_2)\gamma_3-(\gamma_3-\gamma_1)\gamma_2\\
&= (\beta_3+\beta_1)\beta+(\beta_2-\beta)\beta_1\\
&= (\gamma_1+\gamma_2)\gamma+(\gamma_3-\gamma_1)\gamma_1\\
(\beta_2-\beta)(\beta_3+\beta_1) &= (\gamma_1+\gamma_2)\gamma_3-(\gamma_3-\gamma)\gamma_1\\
&= (\alpha_2+\alpha_3)\alpha_1-(\alpha_1-\alpha)\alpha_3\\
&= (\gamma_1+\gamma_2)\gamma+(\gamma_3-\gamma)\gamma_2\\
&= (\alpha_2+\alpha_3)\alpha+(\alpha_1-\alpha)\alpha_2\\
(\gamma_3-\gamma)(\gamma_1+\gamma_2) &= (\alpha_2+\alpha_3)\alpha_1-(\alpha_1-\alpha)\alpha_2\\
&= (\beta_3+\beta_1)\beta_2-(\beta_2-\beta)\beta_1\\
&= (\alpha_2+\alpha_3)\alpha+(\alpha_1-\alpha)\alpha_3\\
&= (\beta_3+\beta_1)\beta+(\beta_2-\beta)\beta_3
\end{aligned}\right\}\quad(40)$$

$$\left.\begin{aligned}
(\alpha_1-\alpha)(\beta_2-\beta)(\gamma_3-\gamma):(\alpha_2+\alpha_3)(\beta_3+\beta_1)(\gamma_1+\gamma_2) &= r:s\\
(\alpha_1-\alpha)(\beta_3+\beta_1)(\gamma_1+\gamma_2):(\alpha_2+\alpha_3)(\beta_2-\beta)(\gamma_3-\gamma) &= r_1:s_1\\
(\alpha_2+\alpha_3)(\beta_2-\beta)(\gamma_1+\gamma_2):(\alpha_1-\alpha)(\beta_3+\beta_1)(\gamma_3-\gamma) &= r_2:s_2\\
(\alpha_2+\alpha_3)(\beta_3+\beta_1)(\gamma_3-\gamma):(\alpha_1-\alpha)(\beta_2-\beta)(\gamma_1+\gamma_2) &= r_3:s_3
\end{aligned}\right\}\quad(41)$$

By combining (41) with (8) and (20) other proportions may be obtained which it is needless to write down. T. S. Davies (in the *Ladies' Diary* for 1835, p. 53) gives one of them:

$$(\alpha_1-\alpha)(\beta_2-\beta)(\gamma_3-\gamma):(\alpha_2+\alpha_3)(\beta_3+\beta_1)(\gamma_1+\gamma_2)=\alpha\beta\gamma:abc \qquad (42)$$

$$\left.\begin{aligned}&(\alpha_2+\alpha_3)(\beta_3+\beta_1)(\gamma_1+\gamma_2)\\&\quad=(\alpha_2+\alpha_3)(\beta_2-\beta)(\gamma_3-\gamma)+(\alpha_1-\alpha)(\beta_3+\beta_1)(\gamma_3-\gamma)\\&\qquad+(\alpha_1-\alpha)(\beta_2-\beta)(\gamma_1+\gamma_2)\end{aligned}\right\} \qquad (43)$$

$$\left.\begin{aligned}&(\alpha_1-\alpha)\alpha s=(\beta_2-\beta)\beta s=(\gamma_3-\gamma)\gamma s\\&=(\alpha_2+\alpha_3)\alpha r_1=(\beta_3+\beta_1)\beta r_2=(\gamma_1+\gamma_2)\gamma r_3\\&=(\alpha_1-\alpha)\alpha_1 s_1=(\beta_2-\beta)\beta_1 r_3=(\gamma_3-\gamma)\gamma_1 r_2\\&=(\alpha_2+\alpha_3)\alpha_1 r=(\beta_3+\beta_1)\beta_1 s_1=(\gamma_1+\gamma_2)\gamma_1 s_1\\&=(\alpha_1-\alpha)\alpha_2 r_3=(\beta_2-\beta)\beta_2 s_2=(\gamma_3-\gamma)\gamma_2 r_1\\&=(\alpha_2+\alpha_3)\alpha_2 s_2=(\beta_3+\beta_1)\beta_2 r=(\gamma_1+\gamma_2)\gamma_2 s_2\\&=(\alpha_1-\alpha)\alpha_3 r_2=(\beta_2-\beta)\beta_3 r_1=(\gamma_3-\gamma)\gamma_3 s_3\\&=(\alpha_2+\alpha_3)\alpha_3 s_3=(\beta_3+\beta_1)\beta_3 s_3=(\gamma_1+\gamma_2)\gamma_3 r_3\\&=abc\end{aligned}\right\} \qquad (44)$$

$$\left.\begin{aligned}&\frac{1}{(\alpha_1-\alpha)^2}+\frac{1}{(\alpha_2+\alpha_3)^2}=\frac{1}{a^2}\\&\frac{1}{(\beta_2-\beta)^2}+\frac{1}{(\beta_3+\beta_1)^2}=\frac{1}{b^2}\\&\frac{1}{(\gamma_3-\gamma)^2}+\frac{1}{(\gamma_1+\gamma_2)^2}=\frac{1}{c^2}\end{aligned}\right\} \qquad (45)$$

$$\left.\begin{aligned}&\frac{\alpha}{\alpha_1}+\frac{\beta}{\beta_2}+\frac{\gamma}{\gamma_3}=1 \qquad &&\frac{\alpha_1}{\alpha}-\frac{\beta_1}{\beta_3}-\frac{\gamma_1}{\gamma_2}=1\\&-\frac{\alpha_2}{\alpha_3}+\frac{\beta_2}{\beta}-\frac{\gamma_2}{\gamma_1}=1 \qquad &&-\frac{\alpha_3}{\alpha_2}-\frac{\beta_3}{\beta_1}+\frac{\gamma_3}{\gamma}=1\end{aligned}\right\} \qquad (46)$$

These equalities are merely particular cases of more general ones stated by Gergonne in his *Annales*, IX., 116, 284 (1818-9).

$$\left.\begin{aligned}
\frac{\alpha_1-\alpha}{\alpha_1}+\frac{\beta_2-\beta}{\beta_2}+\frac{\gamma_3-\gamma}{\gamma_3}&=2\\
-\frac{\alpha_1-\alpha}{\alpha}+\frac{\beta_3+\beta_1}{\beta_3}+\frac{\gamma_1+\gamma_2}{\gamma_2}&=2\\
\frac{\alpha_2+\alpha_3}{\alpha_3}-\frac{\beta_2-\beta}{\beta}+\frac{\gamma_1+\gamma_2}{\gamma_1}&=2\\
\frac{\alpha_2+\alpha_3}{\alpha_2}+\frac{\beta_3+\beta_1}{\beta_1}-\frac{\gamma_3-\gamma}{\gamma}&=2
\end{aligned}\right\} \qquad (47)$$

The first of these equations is a particular case of a theorem given by Vecten in Gergonne's *Annales*, IX., 277–9 (1819).

$$\left.\begin{aligned}
\frac{1}{\alpha^2}+\frac{1}{\alpha_1^2}+\frac{1}{\alpha_2^2}+\frac{1}{\alpha_3^2}&=\frac{4}{h_1^2}\\
\frac{1}{\beta^2}+\frac{1}{\beta_1^2}+\frac{1}{\beta_2^2}+\frac{1}{\beta_3^2}&=\frac{4}{h_2^2}\\
\frac{1}{\gamma^2}+\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}+\frac{1}{\gamma_3^2}&=\frac{4}{h_3^2}
\end{aligned}\right\} \qquad (48)$$

$$\Sigma\left(\frac{1}{\alpha^2}\right)+\Sigma\left(\frac{1}{\beta^2}\right)+\Sigma\left(\frac{1}{\gamma^2}\right)=\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2} \qquad (49)$$

See (35) of the r formulae.*

$$\left.\begin{aligned}
4\Delta_0 &= 2(\alpha_2+\alpha_3)\,\alpha_1 = 2(\beta_3+\beta_1)\,\beta_2 = 2(\gamma_1+\gamma_2)\,\gamma_3\\
&= (\alpha_1-\alpha)(\alpha_2+\alpha_3)+(\beta_2-\beta)(\beta_3+\beta_1)+(\gamma_3-\gamma)(\gamma_1+\gamma_2)\\
4\Delta_1 &= 2(\alpha_2+\alpha_3)\,\alpha = 2(\beta_2-\beta)\,\beta_3 = 2(\gamma_3-\gamma)\,\gamma_2\\
&= -(\alpha_1-\alpha)(\alpha_2+\alpha_3)+(\beta_2-\beta)(\beta_3+\beta_1)+(\gamma_3-\gamma)(\gamma_1+\gamma_2)\\
4\Delta_2 &= 2(\alpha_1-\alpha)\,\alpha_3 = 2(\beta_3+\beta_1)\,\beta = 2(\gamma_3-\gamma)\,\gamma_1\\
&= (\alpha_1-\alpha)(\alpha_2+\alpha_2)-(\beta_2-\beta)(\beta_3+\beta_1)+(\gamma_3-\gamma)(\gamma_1+\gamma_2)\\
4\Delta_3 &= 2(\alpha_1-\alpha)\,\alpha_2 = 2(\beta_2-\beta)\,\beta_1 = 2(\gamma_1+\gamma_2)\,\gamma\\
&= (\alpha_1-\alpha)(\alpha_2+\alpha_3)+(\beta_2-\beta)(\beta_3+\beta_1)-(\gamma_3-\gamma)(\gamma_1+\gamma_2)
\end{aligned}\right\} \qquad (50)$$

where	Δ_0	Δ_1	Δ_2	Δ_3	denote
triangles	$I_1I_2I_3$	II_3I_2	I_3II_1	I_2I_1I.	

* *Proceedings of the Edinburgh Mathematical Society*, Vol. XII., p. 94 (1894).

HISTORICAL NOTES.

In 1841 the *Ladies' Diary*, which first appeared in 1704, and the *Gentleman's Diary*, which first appeared in 1741, were united and published under the title of the *Lady's and Gentleman's Diary*, which came to an end in 1871. This title will in the notes be shortened to *Diary*.

(1) The values of $\alpha_1\ \beta_2\ \gamma_3$ are given by J. Lowry in the *Ladies' Diary* for for 1836, p. 52; T. S. Davies adds six more in the *Diary* for 1842, p. 79; and Weddle completes the dozen by giving the values of $\alpha\ \beta\ \gamma$ in the *Diary* for 1843, p. 80.

(2) C. J. Matthes in his *Commentatio de Proprietatibus Quinque Circulorum*, pp. 46, 49 (1831).

(3) Weddle in the *Diary* for 1843, p. 81.

(4) Lhuilier in his *Élémens d'Analyse*, p. 215 (1809). The values of $\alpha\alpha_1\ \beta\beta_2\ \gamma\gamma_3$ were however given by J. Lowry in Leybourn's *Mathematical Repository*, old series, I. 394 (1799).

(5) T. T. Wilkinson in *Mathematical Questions from the Educational Times*, XIX. 107 (1873).

(6) C. Adams in *Die merkwürdigsten Eigenschaften des geradlinigen Dreiecks*, p. 36 (1846).

(7)—(12) Weddle in the *Diary* for 1843, pp. 81, 82, 88. The first three proportions of (12) are however implicitly given by Matthes in his *Commentatio*, p. 46 (1831).

(13) The first property was proposed for proof at the *Concours Académique de Clermont*, 1875; the others were given by Mr H. Van Aubel in *Nouvelle Correspondance Mathématique*, IV. 364 (1878).

(14), (17) First property given in Todhunter's *Plane Trigonometry*, Chap. XVI., Ex. 37 (1859).

(15), (18) First property given in Hind's *Trigonometry*, 4th ed., pp. 304, 309 (1841).

(19) First property given in a slightly different form by Adams in his *Eigenschaften des...Dreiecks*, p. 40 (1846).

(20) First property given by C. F. A. Jacobi in his *De Triangulorum Rectilineorum Proprietatibus*, p. 10 (1825).

(21) First proportion given by J. Lowry in the *Ladies' Diary* for 1836, p. 52.

(22) First property on the left side given by Adams in his *Eigenschaften des ...Dreiecks*, p. 62 (1846).

(23) The first property was proposed for proof at the *Concours Académique de Clermont*, 1875. A geometrical solution of it occurs in Bourget's *Journal de Mathématiques Élémentaires*, II. 54-5 (1878).

(24)—(26) Weddle in the *Diary* for 1845, p. 69.

(27)—(29) ,, ,, ,, ,, ,, ,, p. 70.

(30) ,, ,, ,, ,, ,, ,, p. 74.

(31)—(35) ,, ,, ,, ,, ,, ,, p. 71.

(36) The first values of $(a_1 - a)^2$, etc., occur in the *Diary* for 1847, pp. 49-50, in answer to a question proposed the previous year by the editor, W. S. B. Woolhouse. The second values are given by Matthes in his *Commentatio*, pp. 53-4 (1831); the third values by Weddle in the *Diary* for 1845, p. 74. The last values of $(a_2 + a_3)^2$, etc., are given by Franz Unferdinger in Grunert's *Archiv*, XXIX., 436 (1857).

(38) Weddle in the *Diary* for 1843, p. 83.

(39), (40) ,, ,, ,, ,, ,, 1845, p. 73.

(41) The first proportion is given by Adams in his *Eigenschaften des...Dreiecks*, p. 34 (1846). All four follow at once from eight expressions given by Weddle in the *Diary* for 1843, p. 82.

(43) Weddle in the *Diary* for 1843, p. 82.

(45) ,, ,, ,, ,, ,, ,, p. 83.

(46) ,, ,, ,, ,, ,, 1845, p. 76.

(47) J. W. Elliott in the *Diary* for 1847, p. 73.

(48) Weddle ,, ,, ,, ,, 1845, p. 75.

(49) ,, ,, ,, ,, ,, 1845, p. 76.

(50) ,, ,, ,, ,, ,, ,, pp. 72, 75.

§ 10.

Formulae connected with the Angular Bisectors of a Triangle limited at their points of intersection with the sides.

The uniliteral notation for these bisectors

$$l_1 \quad l_2 \quad l_3 \quad \lambda_1 \quad \lambda_2 \quad \lambda_3$$

was suggested by T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 77. In the expressions for them it has been assumed that the sides BC CA AB are in decreasing order of magnitude. Hence it will follow that

BL is less than CL, and BL′ is less than CL′
CM is greater than AM, and CM′ is greater than AM′
AN is less than BN, and AN′ is less than BN′.

The assumption "causes the sign of λ_2 (corresponding to the mean side b) to be contrary to those of λ_1 and λ_3. This must be borne in mind, otherwise the symmetry of the expressions in which these functions $(\lambda_1\ \lambda_2\ \lambda_3)$ are involved will not be seen." (Weddle in the *Diary* for 1848, p. 76.)

Two fundamental theorems* regarding two sides of a triangle and the bisectors of the angles between them give the following proportions:

$$b : c = u_2 : u_1 = u_2' : u_1'$$
$$bc = u_1 u_2 + l_1^2 = u_1' u_2' - \lambda_1^2$$

Hence are derived

$$\left.\begin{aligned} b^2 &= u_2^2 + l_1^2 \cdot \frac{u_2}{u_1} = u_2'^2 - \lambda_1^2 \cdot \frac{u_2'}{u_1'}, \\ c^2 &= u_1^2 + l_1^2 \cdot \frac{u_1}{u_2} = u_1'^2 - \lambda_1^2 \cdot \frac{u_1'}{u_2'} \end{aligned}\right\} \qquad (1)$$

Segments of the sides in terms of the sides

$$\left.\begin{aligned} u_1 &= \frac{ca}{b+c} & v_1 &= \frac{ab}{c+a} & w_1 &= \frac{bc}{a+b} \\ u_1' &= \frac{ca}{b-c} & v_1' &= \frac{ab}{a-c} & w_1' &= \frac{bc}{a-b} \\ u_2 &= \frac{ab}{b+c} & v_2 &= \frac{bc}{c+a} & w_2 &= \frac{ca}{a+b} \\ u_2' &= \frac{ab}{b-c} & v_2' &= \frac{bc}{a-c} & w_2' &= \frac{ca}{a-b} \end{aligned}\right\} \qquad (2)$$

$$\left.\begin{aligned} u_1' + u_1 &= u_2' - u_2 = \mathrm{L\,L'} = \frac{2abc}{b^2 - c^2} \\ v_2' + v_2 &= v_1' - v_1 = \mathrm{MM'} = \frac{2abc}{a^2 - c^2} \\ w_1' + w_1 &= w_2' - w_2 = \mathrm{N\,N'} = \frac{2abc}{a^2 - b^2} \end{aligned}\right\} \qquad (3)$$

* The first is Euclid VI. 3 and its extension, which also was known to the Greeks, as is evident from Pappus's *Mathematical Collection*, VII. 39, second proof. The first part of the second fundamental theorem is given in Schooten's *Exercitationes Mathematicae*, p. 65 (1657).

$$\frac{1}{LL'} - \frac{1}{MM'} + \frac{1}{NN'} = 0 \tag{4}$$

$$\frac{a^2}{LL'} - \frac{b^2}{MM'} + \frac{c^2}{NN'} = 0 \tag{5}$$

$$\frac{a}{LL'} - \frac{b}{MM'} + \frac{c}{NN'} = \frac{(b-c)(c-a)(a-b)}{2abc} \tag{6}$$

The segments of the sides in terms of each other.

$$\left.\begin{aligned} &u_1 = u_1'\frac{u_2' - u_1'}{u_2' + u_1'} \qquad u_2 = u_2'\frac{u_2' - u_1'}{u_2' + u_1'} \\ &\text{and so on.} \\ &u_1' = u_1\frac{u_2 + u_1}{u_2 - u_1} \qquad u_2' = u_2\frac{u_2 + u_1}{u_2 - u_1} \\ &\text{and so on.} \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} &u_1' + u_1 = u_2' - u_2 = LL' = \frac{2u_1u_2}{u_2 - u_1} \\ &\text{and so on.} \end{aligned}\right\} \tag{8}$$

$$LL'^2 = l_1^2 + \lambda_1^2 \qquad MM'^2 = l_2^2 + \lambda_2^2 \qquad NN'^2 = l_3^2 + \lambda_3^2 \tag{9}$$

$$\left.\begin{aligned} l_1 &= \frac{2\sqrt{bcss_1}}{(b+c)} = \frac{2\Delta\sqrt{bcrr_1}}{(b+c)rr_1} \\ l_2 &= \frac{2\sqrt{cass_2}}{(c+a)} = \frac{2\Delta\sqrt{carr_2}}{(c+a)rr_2} \\ l_3 &= \frac{2\sqrt{abss_3}}{(a+b)} = \frac{2\Delta\sqrt{abrr_3}}{(a+b)rr_3} \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned}
\lambda_1 &= \frac{2\sqrt{bcs_2s_3}}{(b-c)} = \frac{2\Delta\sqrt{bcr_2r_3}}{(b-c)r_2r_3} \\
\lambda_2 &= \frac{2\sqrt{cas_3s_1}}{(a-c)} = \frac{2\Delta\sqrt{car_3r_1}}{(a-c)r_3r_1} \\
\lambda_3 &= \frac{2\sqrt{abs_1s_2}}{(a-b)} = \frac{2\Delta\sqrt{abr_1r_2}}{(a-b)r_1r_2}
\end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned}
\frac{l_1^2}{bc} + \frac{a^2}{(b+c)^2} &= 1 \\
\frac{l_2^2}{ca} + \frac{b^2}{(c+a)^2} &= 1 \\
\frac{l_3^2}{ab} + \frac{c^2}{(a+b)^2} &= 1
\end{aligned}\right\} \quad (12) \qquad
\left.\begin{aligned}
\frac{a^2}{(b-c)^2} - \frac{\lambda_1^2}{bc} &= 1 \\
\frac{b^2}{(a-c)^2} - \frac{\lambda_2^2}{ca} &= 1 \\
\frac{c^2}{(a-b)^2} - \frac{\lambda_3^2}{ab} &= 1
\end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned}
l_1^2(b+c)^2 + \lambda_1^2(b-c)^2 &= 4b^2c^2 \\
l_2^2(c+a)^2 + \lambda_2^2(a-c)^2 &= 4c^2a^2 \\
l_3^2(a+b)^2 + \lambda_3^2(a-b)^2 &= 4a^2b^2
\end{aligned}\right\} \quad (14)$$

$$\left.\begin{aligned}
\frac{l_1^2}{bc}(b+c)^2 + \frac{l_2^2}{ca}(c+a)^2 + \frac{l_3^2}{ab}(a+b)^2 &= 4s^2 \\
\frac{l_1^2}{bc}(b+c)^2 - \frac{\lambda_2^2}{ca}(a-c)^2 - \frac{\lambda_3^2}{ab}(a-b)^2 &= 4s_1^2 \\
-\frac{\lambda_1^2}{bc}(b-c)^2 + \frac{l_2^2}{ca}(c+a)^2 - \frac{\lambda_3^2}{ab}(a-b)^2 &= 4s_2^2 \\
-\frac{\lambda_1^2}{bc}(b-c)^2 - \frac{\lambda_2^2}{ca}(a-c)^2 + \frac{l_3^2}{ab}(a+b)^2 &= 4s_3^2
\end{aligned}\right\} \quad (15)$$

$$l_1^2bc\left(\frac{1}{b}+\frac{1}{c}\right)^2 + l_2^2ca\left(\frac{1}{c}+\frac{1}{a}\right)^2 + l_3^2ab\left(\frac{1}{a}+\frac{1}{b}\right)^2 = 4s^2 \quad (16)$$

and so on.

$$\left.\begin{aligned}
\frac{l_1^2}{a^2}\cdot\frac{1}{bc}\left(\frac{1}{b}+\frac{1}{c}\right)^2 + \frac{l_2^2}{b^2}\cdot\frac{1}{ca}\left(\frac{1}{c}+\frac{1}{a}\right)^2 + \frac{l_3^2}{c^2}\cdot\frac{1}{ab}\left(\frac{1}{a}+\frac{1}{b}\right)^2 \\
= \left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)^2
\end{aligned}\right\} \quad (17)$$

and so on.

$$\left(\frac{1}{b}+\frac{1}{c}\right)\frac{b^2-c^2}{l_2^2 l_3^2}+\left(\frac{1}{c}+\frac{1}{a}\right)\frac{c^2-a^2}{l_3^2 l_1^2}+\left(\frac{1}{a}+\frac{1}{b}\right)\frac{a^2-b^2}{l_1^2 l_2^2}=0 \qquad (18)$$

$$\frac{\frac{1}{b^2}-\frac{1}{c^2}}{s_2 s_3}\cdot\frac{b+c}{l_2^2 l_3^2}+\frac{\frac{1}{c^2}-\frac{1}{a^2}}{s_3 s_1}\cdot\frac{c+a}{l_3^2 l_1^2}+\frac{\frac{1}{a^2}-\frac{1}{b^2}}{s_1 s_2}\cdot\frac{a+b}{l_1^2 l_2^2}=0 \qquad (19)$$

$$\left.\begin{aligned} u_1 u_2+v_1 v_2+w_1 w_2 &= abc\left\{\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\right\} \\ u_1' u_2'+v_1' v_2'+w_1' w_2' &= abc\left\{\frac{a}{(b-c)^2}+\frac{b}{(a-c)^2}+\frac{c}{(a-b)^2}\right\} \end{aligned}\right\} \qquad (20)$$

$$\left.\begin{aligned} u_1 u_2+v_1 v_2+w_1 w_2+(l_1^2+l_2^2+l_3^2) &= bc+ca+ab \\ u_1' u_2'+v_1' v_2'+w_1' w_2'-(\lambda_1^2+\lambda_2^2+\lambda_3^2) &= bc+ca+ab \end{aligned}\right\} \qquad (21)$$

$$l_1\alpha+l_2\beta+l_3\gamma=av_1+bw_1+cu_1 \qquad (22)$$

$$\left.\begin{aligned} & l_1(l_1-\alpha)+l_2(l_2-\beta)+l_3(l_3-\gamma) \\ & \qquad =(a-v_1)v_2+(b-w_1)w_2+(c-u_1)u_2 \end{aligned}\right\} \qquad (23)$$

$$\left.\begin{aligned} & \alpha^2+\beta^2+\gamma^2-\{(l_1-\alpha)^2+(l_2-\beta)^2+(l_3-\gamma)^2\} \\ & \qquad =(u_1+v_1+w_1)(u_2+v_2+w_2)-2(u_1 v_2+v_1 w_2+w_1 u_2) \end{aligned}\right\} \qquad (24)$$

$$\left.\begin{aligned} \frac{1}{u_1 v_1 w_1} &= \left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{a}\right)\left(\frac{1}{a}+\frac{1}{b}\right) \\ \frac{1}{u_1 v_1' w_1'} &= \left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{a}\right) \\ \frac{1}{u_1' v_1 w_1'} &= \left(\frac{1}{c}-\frac{1}{b}\right)\left(\frac{1}{c}+\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{a}\right) \\ \frac{1}{u_1' v_1' w_1} &= \left(\frac{1}{c}-\frac{1}{b}\right)\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{a}+\frac{1}{b}\right) \end{aligned}\right\} \qquad (25)$$

These may be put into the forms

$$u_1 v_1 w_1 : abc = abc : (b+c)(c+a)(a+b)$$

and so on; or

$$\mathrm{BL}\cdot\mathrm{CM}\cdot\mathrm{AN} : abc = abc : \mathrm{D_2D_3}\cdot\mathrm{E_3E_1}\cdot\mathrm{F_1F_2}$$

and so on.

$$\left.\begin{aligned}
u_1 v_1 w_1 = u_2 v_2 w_2 &= \frac{4\Delta Rr}{h_1 + h_2 + h_3 - r} \\
u_1 v_1' w_1' = u_2 v_2' w_2' &= \frac{4\Delta Rr_1}{h_1 - h_2 - h_3 + r_1} \\
u_1' v_1 w_1' = u_2' v_2 w_2' &= \frac{4\Delta Rr_2}{h_1 - h_2 + h_3 - r_2} \\
u_1' v_1' w_1 = u_2' v_2' w_2 &= \frac{4\Delta Rr_3}{-h_1 - h_2 + h_3 + r_3}
\end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned}
u_1' v_1' w_1' = u_2' v_2' w_2' &= \frac{a^2b^2c^2}{(b-c)(a-c)(a-b)} \\
u_1' v_1 w_1 = u_2' v_2 w_2 &= \frac{a^2b^2c^2}{(b-c)(c+a)(a+b)} \\
u_1 v_1' w_1 = u_2 v_2' w_2 &= \frac{a^3b^2c^2}{(b+c)(a-c)(a+b)} \\
u_1 v_1 w_1' = u_2 v_2 w_2' &= \frac{a^2b^2c^2}{(b+c)(c+a)(a-b)}
\end{aligned}\right\} \quad (27)$$

These may be put into the forms

$$BL' \cdot CM' \cdot AN' : abc = abc : DD_1 \cdot EE_2 \cdot FF_3$$

and so on.

$$\left.\begin{aligned}
l_1 l_2 l_3 &= \frac{8abcs\Delta}{(b+c)(c+a)(a+b)} & \lambda_1\lambda_2\lambda_3 &= \frac{8abcr\Delta}{(b-c)(a-c)(a-b)} \\
l_1\lambda_2\lambda_3 &= \frac{8abcs_1\Delta}{(b+c)(a-c)(a-b)} & \lambda_1 l_2 l_3 &= \frac{8abcr_1\Delta}{(b-c)(c+a)(a+b)} \\
\lambda_1 l_2 \lambda_3 &= \frac{8abcs_2\Delta}{(b-c)(c+a)(a-b)} & l_1 \lambda_2 l_3 &= \frac{8abcr_2\Delta}{(b+c)(a-c)(a+b)} \\
\lambda_1\lambda_2 l_3 &= \frac{8abcs_3\Delta}{(b-c)(a-c)(a+b)} & l_1 l_2 \lambda_3 &= \frac{8abcr_3\Delta}{(b+c)(c+a)(a-b)}
\end{aligned}\right\} \quad (28)$$

$$l_1 l_2 l_3 = \frac{32R\Delta^3}{r(b+c)(c+a)(a+b)} = \frac{8\Delta^2}{h_1 + h_2 + h_3 - r} \quad (29)$$

and so on.

$$\left.\begin{array}{l}\lambda_1\lambda_2\lambda_3 = \dfrac{32R\Delta}{s(b-c)(a-c)(a-b)} \\ \text{and so on.}\end{array}\right\} \qquad (30)$$

$$BL \cdot CM \cdot AN : l_1 l_2 l_2 = R : 2s \qquad (31)$$

$$BL' \cdot CM' \cdot AN' : \lambda_1\lambda_2\lambda_3 = R : 2r \qquad (32)$$

$$\left.\begin{array}{l} l_1 l_2 l_3 (b+c)(c+a)(a+b) \\ = 8\alpha\,\beta\,\gamma\,s^3 = 8\alpha_1\beta_1\gamma_1 s s_1^2 = 8\alpha_2\beta_2\gamma_2 s s_2^2 = 8\alpha_3\beta_3\gamma_3 s s_3^2 \\ = 8\alpha_1\beta_2\gamma_3 s r^2 = 8\alpha\,\beta_3\gamma_2 s r_1^2 = 8\alpha_3\beta\,\gamma_1 s r_2^2 = 8\alpha_2\beta_1\gamma\,s r_3^2 \end{array}\right\} \qquad (33)$$

$$\left.\begin{array}{l} \lambda_1\lambda_2\lambda_3(b-c)(a-c)(a-b) \\ = 8\alpha_1\beta_2\gamma_3 r^3 = 8\alpha\,\beta_3\gamma_2 r r_1^2 = 8\alpha_3\beta\,\gamma_1 r r_2^2 = 8\alpha_2\beta_1\gamma\,r r_3^2 \\ = 8\alpha\,\beta\,\gamma\,r s^2 = 8\alpha_1\beta_1\gamma_1 r s_1^2 = 8\alpha_2\beta_2\gamma_2 r s_2^2 = 8\alpha_3\beta_3\gamma_3 r s_3^2 \end{array}\right\} \qquad (34)$$

$$\left.\begin{array}{l} l_1 l_2 l_3 \lambda_1\lambda_2\lambda_3 = \dfrac{64a^2b^2c^2\Delta^3}{(b^2-c^2)(a^2-c^2)(a^2-b^2)} \\ \qquad = \dfrac{1024R^2\Delta^5}{(b^2-c^2)(a^2-c^2)(a^2-b^2)} \end{array}\right\} \qquad (35)$$

$$\frac{2a}{l_1\lambda_1} = \frac{h_3^2 - h_2^2}{h_1 h_2 h} \qquad \frac{2b}{l_2\lambda_2} = \frac{h_3^2 - h_1^2}{h_1 h_2 h_3} \qquad \frac{2c}{l_3\lambda_3} = \frac{h_2^2 - h_1^2}{h_1 h_2 h_3} \qquad (36)$$

$$\frac{2h_1}{l_1\lambda_1} = \frac{b^2 - c^2}{abc} \qquad \frac{2h_2}{l_2\lambda_2} = \frac{a^2 - c^2}{abc} \qquad \frac{2h_3}{l_3\lambda_3} = \frac{a^2 - b^2}{abc} \qquad (37)$$

$$l_1\lambda_1 = \frac{4bc\Delta}{b^2 - c^2} \qquad l_2\lambda_2 = \frac{4ca\Delta}{a^2 - c^2} \qquad l_3\lambda_3 = \frac{4ab\Delta}{a^2 - b^2} \qquad (38)$$

$$\frac{a}{l_1\lambda_1} - \frac{b}{l_2\lambda_2} + \frac{c}{l_3\lambda_3} = 0 \qquad (39)$$

$$\frac{h_1}{l_1\lambda_1} - \frac{h_2}{l_2\lambda_2} + \frac{h_3}{l_3\lambda_3} = 0 \qquad (40)$$

$$\frac{1}{al_1\lambda_1} - \frac{1}{bl_2\lambda_2} + \frac{1}{cl_3\lambda_3} = 0 \qquad (41)$$

$$\frac{1}{h_1l_1\lambda_1} - \frac{1}{h_2l_2\lambda_2} + \frac{1}{h_3l_3\lambda_3} = 0 \qquad (42)$$

$$\left.\begin{aligned}
\alpha &: l_1 = r\,(b+c) : 2\Delta = b+c : 2s \\
\beta &: l_2 = r\,(c+a) : 2\Delta = c+a : 2s \\
\gamma &: l_3 = r\,(a+b) : 2\Delta = a+b : 2s \\[4pt]
\alpha_1 &: l_1 = r_1(b+c) : 2\Delta = b+c : 2s_1 \\
\beta_2 &: l_2 = r_2(c+a) : 2\Delta = c+a : 2s_2 \\
\gamma_3 &: l_3 = r_3(a+b) : 2\Delta = a+b : 2s_3 \\[4pt]
\alpha_2 &: \lambda_1 = r_2(b-c) : 2\Delta = b-c : 2s_2 \\
\alpha_3 &: \lambda_1 = r_3(b-c) : 2\Delta = b-c : 2s_3 \\[4pt]
\beta_3 &: \lambda_2 = r_3(a-c) : 2\Delta = a-c : 2s_3 \\
\beta_1 &: \lambda_2 = r_1(a-c) : 2\Delta = a-c : 2s_1 \\[4pt]
\gamma_1 &: \lambda_3 = r_1(a-b) : 2\Delta = a-b : 2s_1 \\
\gamma_2 &: \lambda_3 = r_3(a-b) : 2\Delta = a-b : 2s_2
\end{aligned}\right\} \qquad (43)$$

$$\left.\begin{aligned}
\alpha &: \mathrm{I\,L} = b+c : a \\
\beta &: \mathrm{I\,M} = c+a : b \\
\gamma &: \mathrm{I\,N} = a+b : c \\[4pt]
\alpha_1 &: \mathrm{I_1L} = b+c : a \\
\beta_2 &: \mathrm{I_2M} = c+a : b \\
\gamma_3 &: \mathrm{I_3N} = a+b : c \\[4pt]
\alpha_2 &: \mathrm{I_2L'} = b-c : a \\
\alpha_3 &: \mathrm{I_3L'} = b-c : a \\[4pt]
\beta_3 &: \mathrm{I_3M'} = a-c : b \\
\beta_1 &: \mathrm{I_1M'} = a-c : b \\[4pt]
\gamma_1 &: \mathrm{I_1N'} = a-b : c \\
\gamma_2 &: \mathrm{I\,N'} = a-b : c
\end{aligned}\right\} \ (44)
\qquad
\left.\begin{aligned}
l_1 &: \mathrm{I\,L} = 2s : a \\
l_2 &: \mathrm{I\,M} = 2s : b \\
l_3 &: \mathrm{I\,N} = 2s : c \\[4pt]
l_1 &: \mathrm{I_1L} = 2s_1 : a \\
l_2 &: \mathrm{I_2M} = 2s_2 : b \\
l_3 &: \mathrm{I_3N} = 2s_3 : c \\[4pt]
\lambda_1 &: \mathrm{I_2L'} = 2s_2 : a \\
\lambda_1 &: \mathrm{I_3L'} = 2s_3 : a \\[4pt]
\lambda_2 &: \mathrm{I_3M'} = 2s_3 : b \\
\lambda_2 &: \mathrm{I_1M'} = 2s_1 : b \\[4pt]
\lambda_3 &: \mathrm{I_1N'} = 2s_1 : c \\
\lambda_3 &: \mathrm{I_2N'} = 2s_2 : c
\end{aligned}\right\} \ (45)$$

Values such as

$$\left.\begin{aligned}
l_1 &= \frac{2\Delta}{r}\cdot\frac{a}{D_2D_3} = \frac{2\Delta}{r_1}\cdot\frac{a_1}{D_2D_3}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\lambda_1 &= \frac{2\Delta}{r_2}\cdot\frac{a_2}{DD_1} = \frac{2\Delta}{r_3}\cdot\frac{a_3}{DD_1}\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\tag{46}$$

$$\left.\begin{aligned}
IL &= \frac{a\sqrt{bcrr_1}}{(b+c)r_1} \qquad I_1L = \frac{a\sqrt{bcrr_1}}{(b+c)r}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
I_2L' &= \frac{a\sqrt{bcr_2r_3}}{(b-c)r_3} \qquad I_3L' = \frac{a\sqrt{bcr_2r_3}}{(b-c)r_2}\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\tag{47}$$

need not be written out at length.

$$\left.\begin{aligned}
IL\cdot IM\cdot IN &= \frac{16\Delta R^2r^2}{(b+c)(c+a)(a+b)}\\
I_1L\cdot I_2M\cdot I_3N &= \frac{16\Delta R^2s^2}{(b+c)(c+a)(a+b)}\\
I_1L\cdot I_1M'\cdot I_1N' &= \frac{16\Delta R^2r_1^2}{(b+c)(a-c)(a-b)}\\
IL\cdot I_3M'\cdot I_2N' &= \frac{16\Delta R^2s_1^2}{(b+c)(a-c)(a-b)}\\
I_2L'\cdot I_2M\cdot I_2N' &= \frac{16\Delta R^2r_2^2}{(b-c)(c+a)(a-b)}\\
I_3L'\cdot IM\cdot I_1N' &= \frac{16\Delta R^2s_2^2}{(b-c)(c+a)(a-b)}\\
I_3L'\cdot I_3M'\cdot I_3N &= \frac{16\Delta R^2r_3^2}{(b-c)(a-c)(a+b)}\\
I_2L'\cdot I_1M'\cdot IN &= \frac{16\Delta R^2s_3^2}{(b-c)(a-c)(a+b)}
\end{aligned}\right\}\tag{48}$$

$$\left.\begin{aligned} \mathrm{IL}\cdot\mathrm{IM}\cdot\mathrm{IN} &= \frac{4\mathrm{R}r^3}{h_1+h_2+h_3-r} \\ \mathrm{I_1L}\cdot\mathrm{I_2M}\cdot\mathrm{I_3N} &= \frac{4\mathrm{R}rs^2}{h_1+h_2+h_3-r} \end{aligned}\right\} \quad (49)$$

$$\left.\begin{aligned} \alpha\beta\gamma : \mathrm{IL}\cdot\mathrm{IM}\cdot\mathrm{IN} &= (b+c)(c+a)(a+b) : abc \\ &= abc : \mathrm{BL}\cdot\mathrm{CM}\cdot\mathrm{AN} \\ &= h_1+h_2+h_3-r : r \\ &= \alpha_1\beta_2\gamma_3\cdot\mathrm{I_1L} : \mathrm{I_2M}\cdot\mathrm{I_3N} \end{aligned}\right\} \quad (50)$$

$$\left.\begin{aligned} \mathrm{IL}\cdot\mathrm{IM}\cdot\mathrm{IN} : l_1l_2l_3 &= \mathrm{R}r : 2s^2 \\ \mathrm{I_1L}\cdot\mathrm{I_2M}\cdot\mathrm{I_3N} : l_1l_2l_3 &= \mathrm{R} \ \ : 2r \end{aligned}\right\} \quad (51)$$

$$\left.\begin{aligned} \frac{l_1}{\mathrm{LL'}}\cdot\frac{l_2}{\mathrm{MM'}}\cdot\frac{l_3}{\mathrm{NN'}} &= \frac{(b-c)(a-c)(a-b)}{16\mathrm{R}^2r} = \frac{h_1h_2h_3}{\lambda_1\lambda_2\lambda_3} \\ \frac{\lambda_1}{\mathrm{LL'}}\cdot\frac{\lambda_2}{\mathrm{MM'}}\cdot\frac{\lambda_3}{\mathrm{NN'}} &= \frac{(b+c)(c+a)(a+b)}{16\mathrm{R}^2s} = \frac{h_1h_2h_3}{l_1l_2l_3} \end{aligned}\right\} \quad (52)$$

$$\left.\begin{aligned} l_1^2 &= \frac{4rr_1(rr_1+r_2r_3)}{(r_1+r)^2} & \lambda_1^2 &= \frac{4r_2r_3(rr_1+r_2r_3)}{(r_2-r_3)^2} \\ l_2^2 &= \frac{4rr_2(rr_2+r_3r_1)}{(r_2+r)^2} & \lambda_2^2 &= \frac{4r_3r_1(rr_2+r_3r_1)}{(r_1-r_3)^2} \\ l_3^2 &= \frac{4rr_3(rr_3+r_1r_2)}{(r_3+r)^2} & \lambda_3^2 &= \frac{4r_1r_2(rr_3+r_1r_2)}{(r_1-r_2)} \end{aligned}\right\} \quad (53)$$

$$\left.\begin{aligned} \frac{1}{l_1^2}+\frac{1}{\lambda_1^2} &= \frac{1}{h_1^2} \\ \frac{1}{l_2^2}+\frac{1}{\lambda_2^2} &= \frac{1}{h_2^2} \\ \frac{1}{l_3^2}+\frac{1}{\lambda_3^2} &= \frac{1}{h_3^2} \end{aligned}\right\} \quad (54)$$

$$\left.\begin{aligned}
\frac{l_1}{l_2l_3}+\frac{l_1}{\lambda_2\lambda_3}&=\frac{\lambda_1}{\lambda_2l_3}-\frac{\lambda_1}{l_2\lambda_3}=\frac{h_1}{h_2h_3}=\frac{2R}{a^2}\\
\frac{l_2}{l_3l_1}-\frac{l_2}{\lambda_3\lambda_1}&=\frac{\lambda_2}{l_3\lambda_1}+\frac{\lambda_2}{\lambda_3l_1}=\frac{h_2}{h_3h_1}=\frac{2R}{b^2}\\
\frac{l_3}{l_1l_2}+\frac{l_3}{\lambda_1\lambda_2}&=\frac{\lambda_3}{l_1\lambda_2}-\frac{\lambda_3}{\lambda_1l_2}=\frac{h_3}{h_1h_2}=\frac{2R}{c^2}
\end{aligned}\right\}\tag{55}$$

$$\left.\begin{aligned}
\frac{l_1}{\lambda_1}&=\frac{l_2\lambda_3-\lambda_2l_3}{l_2l_3+\lambda_2\lambda_3}\\
\frac{l_2}{\lambda_2}&=\frac{l_3\lambda_1+\lambda_3l_1}{\lambda_3\lambda_1-l_3l_1}\\
\frac{l_3}{\lambda_3}&=\frac{\lambda_1l_2-l_1\lambda_2}{l_1l_2+\lambda_1\lambda_2}
\end{aligned}\right\}\tag{56}$$

Weddle remarks that the three preceding relations between $l_1\ l_2\ l_3\ \lambda_1\ \lambda_2\ \lambda_3$ all reduce to

$$l_1l_2l_3=\lambda_1l_2\lambda_3-\lambda_1\lambda_2l_3-l_1\lambda_2\lambda_3\tag{57}$$

$$\left.\begin{aligned}
&l_1=\frac{2\alpha\alpha_1}{\alpha+\alpha_1} &&l_2=\frac{2\beta\beta_2}{\beta+\beta_2} &&l_3=\frac{2\gamma\gamma_3}{\gamma+\gamma_3}\\
&\frac{2}{l_1}=\frac{1}{\alpha}+\frac{1}{\alpha_1} &&\frac{2}{l_2}=\frac{1}{\beta}+\frac{1}{\beta_2} &&\frac{2}{l_3}=\frac{1}{\gamma}+\frac{1}{\gamma_3}\\
&\lambda_1=\frac{2\alpha_2\alpha_3}{\alpha_2-\alpha_3} &&\lambda_2=\frac{2\beta_1\beta_3}{\beta_1-\beta_3} &&\lambda_3=\frac{2\gamma_1\gamma_2}{\gamma_1-\gamma_2}\\
&\frac{2}{\lambda_1}=\frac{1}{\alpha_3}-\frac{1}{\alpha_2} &&\frac{2}{\lambda_2}=\frac{1}{\beta_3}-\frac{1}{\beta_1} &&\frac{2}{\lambda_3}=\frac{1}{\gamma_2}-\frac{1}{\gamma_1}
\end{aligned}\right\}\tag{58}$$

$$\left.\begin{aligned}
&2AU=\alpha_1+\alpha &&2BV=\beta_2+\beta &&2CW=\gamma_3+\gamma\\
&2AU'=\alpha_2-\alpha_3 &&2BV'=\beta_1-\beta_3 &&2CW'=\gamma_1-\gamma_2
\end{aligned}\right\}\tag{59}$$

$$\left.\begin{aligned}
AU\,(\alpha_2+\alpha_3) &= 2R(b+c)\\
BV\,(\beta_3+\beta_1) &= 2R(c+a)\\
CW\,(\gamma_1+\gamma_2) &= 2R(a+b)\\
AU'(\alpha_1-\alpha\,) &= 2R(b-c)\\
BV'(\beta_2-\beta\,) &= 2R(a-c)\\
CW'(\gamma_3-\gamma\,) &= 2R(a-b)
\end{aligned}\right\} \quad (60)$$

$$\left.\begin{aligned}
AU\cdot l_1 &= AU'\cdot\lambda_1 = bc\\
BV\cdot l_2 &= BV'\cdot\lambda_2 = ca\\
CW\cdot l_3 &= CW'\cdot\lambda_3 = ab
\end{aligned}\right\} \quad (61)$$

$$\left.\begin{aligned}
AU^2 &= \frac{bc(b+c)^2}{4ss_1} \qquad & AU'^2 &= \frac{bc(b-c)^2}{4s_2s_3}\\
BV^2 &= \frac{ca(c+a)^2}{4ss_2} \qquad & BV'^2 &= \frac{ac(a-c)^2}{4s_3s_1}\\
CW^2 &= \frac{ab(a+b)^2}{4ss_3} \qquad & CW'^2 &= \frac{ab(a-b)^2}{4s_1s_2}
\end{aligned}\right\} \quad (62)$$

$$\left.\begin{aligned}
UA' &= \frac{r^2}{h_1-2r} = \frac{r_1}{h_1+2r_1}\\
VB' &= \frac{r^2}{h_2-2r} = \frac{r_2^{\,2}}{h_2+2r_2}\\
WC' &= \frac{r^2}{h_3-2r} = \frac{r_3^{\,2}}{h_3+2r_3}
\end{aligned}\right\} \quad (63)$$

$$\frac{bc(b^2-c^2)}{4AU\cdot AU'} = \frac{ac(a^2-c^2)}{4BV\cdot BV'} = \frac{ab(a^2-b^2)}{4CW\cdot CW'} = \Delta \qquad (64)$$

$$\left.\begin{aligned}
l_1(b+c) &= h_1(\alpha_2+\alpha_3) \qquad & \lambda_1(b-c) &= h_1(\alpha_1-\alpha)\\
l_2(c+a) &= h_2(\beta_3+\beta_1) \qquad & \lambda_2(a-c) &= h_2(\beta_2-\beta)\\
l_3(a+b) &= h_3(\gamma_1+\gamma_2) \qquad & \lambda_3(a-b) &= h_3(\gamma_3-\gamma)
\end{aligned}\right\} \quad (65)$$

$$\left.\begin{aligned}
\frac{4}{l_1^2}+\frac{4}{\lambda_1^2}&=\frac{1}{a^2}+\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}\\
\frac{4}{l_2^2}+\frac{4}{\lambda_2^2}&=\frac{1}{\beta^2}+\frac{1}{\beta_1^2}+\frac{1}{\beta_2^2}+\frac{1}{\beta_3^2}\\
\frac{4}{l_3^2}+\frac{4}{\lambda_3^2}&=\frac{1}{\gamma^2}+\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}+\frac{1}{\gamma_3^2}
\end{aligned}\right\}\quad(66)$$

$$\left.\begin{aligned}
\frac{a}{l_1}+\frac{\beta}{l_2}+\frac{\gamma}{l_3}&=2\\
\frac{a_1}{l_1}-\frac{\beta_1}{\lambda_2}+\frac{\gamma_1}{\lambda_3}&=2\\
\frac{\beta_2}{l_2}+\frac{\gamma_2}{\lambda_3}-\frac{a_2}{\lambda_1}&=2\\
\frac{\gamma_3}{l_3}+\frac{a_3}{\lambda_1}+\frac{\beta_3}{\lambda_2}&=2
\end{aligned}\right\}\quad(67)
\qquad
\left.\begin{aligned}
\frac{a_1}{\lambda_1}-\frac{\beta_2}{\lambda_2}+\frac{\gamma_3}{\lambda_3}&=0\\
\frac{a}{\lambda_1}+\frac{\beta_3}{l_2}-\frac{\gamma_2}{l_3}&=0\\
\frac{a_3}{l_1}+\frac{\beta}{\lambda_2}-\frac{\gamma_1}{l_3}&=0\\
\frac{a_2}{l_1}-\frac{\beta_1}{l_2}+\frac{\gamma}{\lambda_3}&=0
\end{aligned}\right\}\quad(68)$$

Let AI BI CI meet MN NL LM
respectively at L_1 M_1 N_1

$$\left.\begin{aligned}
AL_1 : IL_1 &= AL : IL = h_1 : r\\
BM_1 : IM_1 &= BM : IM = h_2 : r\\
CN_1 : IN_1 &= CN : IN = h_3 : r
\end{aligned}\right\}\quad(69)$$

$$\left.\begin{aligned}
AL_1&=\frac{h_1 a}{h_1+r} & BM_1&=\frac{h_2\beta}{h_2+r} & CN_1&=\frac{h_3\gamma}{h_3+r}\\
IL_1&=\frac{ra}{h_1+r} & IM_1&=\frac{r\beta}{h_2+r} & IN_1&=\frac{r\gamma}{h_3+r}
\end{aligned}\right\}\quad(70)$$

Matthes (p. 47) gives the values

$$\left.\begin{aligned}
AL_1&=\frac{h_1\sqrt{bcrr_1}}{(h_1+r)r_1}\text{, etc.,}\\
IL_1&=\frac{r\sqrt{bcrr_1}}{(h_1+r)r_1}\text{, etc.}
\end{aligned}\right\}\quad(71)$$

Expressions for the sides of $\triangle$LMN.

$$\left.\begin{array}{l} \mathrm{MN}^2 = \dfrac{abc}{(c+a)^2(a+b)^2} \times \\ (b^2c + bc^2 - c^2a + ca^2 + a^2b - ab^2 + a^3 - b^2 - c^3 + 3abc) \end{array}\right\} \quad (72)$$

NL^2 and LM^2 can be obtained by cyclical permutations of the letters $a\ b\ c$.

These expressions can be put into shorter forms, by help of Landen's theorem that

$$\mathrm{I_1O}^2 = \mathrm{R}^2 + 2\mathrm{R}r_1$$

For
$$4\triangle(\mathrm{R} + 2r_1) = 4\triangle\mathrm{R} + \frac{16\triangle^2}{2s_1}$$
$$= abc + (a+b+c)(a-b+c)(a+b-c)$$
$$= b^2c + bc^2 - c^2a + ca^2 + a^2b - ab^2 + a^3 - b^3 - c^3 + 3abc$$

Hence

$$\mathrm{MN} = \frac{4\triangle \cdot \mathrm{I_1O}}{(c+a)(a+b)} \qquad \mathrm{NL} = \frac{4\triangle \cdot \mathrm{I_2O}}{(a+b)(b+c)} \qquad \mathrm{LM} = \frac{4\triangle \cdot \mathrm{I_3O}}{(b+c)(c+a)} \quad (73)$$

Matthes (p. 45) in transforming the ten-term factor which occurs in the expression for MN^2 does not appear to have observed the simplification that would result from introducing $\mathrm{R} + 2r_1$. He introduces $\mathrm{R} + 2\mathrm{r}$, and obtains for MN the following value:

$$\frac{4\triangle}{(c+a)(a+b)r_2r_3}\sqrt{\{(\mathrm{R}^2 + 2\mathrm{R}r)r_2r_3 + 2a\triangle\mathrm{R}\}r_2r_3}$$

The points L′ M′ N′ do not form the vertices of a triangle, but are collinear.

Expressions for the distances M′N′ N′L′ L′M′.

$$\left.\begin{array}{l} \mathrm{M'N'}^2 = \dfrac{abc}{(a-c)^2(a-b)^2} \times \\ (-b^2c - bc^2 - c^2a - ca^2 - a^2b - ab^2 + a^3 + b^3 + c^3 + 3abc) \end{array}\right\} \quad (74)$$

$\mathrm{N'L'}^2$ and $\mathrm{L'M'}^2$ can be obtained by cyclical permutations of the letters $a\ b\ c$.

These expressions can be put into shorter forms, by help of Chapple's theorem that

$$IO^2 = R^2 - 2Rr$$

For $$4\Delta(R - 2r) = 4\Delta R - \frac{16\Delta^2}{2s}$$

$$= abc + (-a+b+c)(a-b+c)(a+b-c)$$

$$= -b^2c - bc^2 - c^2a - ca^2 - a^2b - ab^2 + a^3 + b^3 + c^3 + 3abc$$

Hence

$$M'N' = \frac{4\Delta \cdot IO}{(a-c)(a-b)} \quad N'L' = \frac{4\Delta \cdot IO}{(a-b)(b-c)} \quad L'M' = \frac{4\Delta \cdot IO}{(b-c)(a-c)} \quad (75)$$

In deducing the expressions for $M'N'$ $N'L'$ $L'M'$ it has been assumed that a b c are in descending order of magnitude. If the figure do not correspond to this supposition, care must be taken in verifying the equation

$$L'M' = M'N' + N'L'$$

to affix the proper signs to the values of these magnitudes.

Historical Notes.

(3) Crelle's *Eigenschaften des...Dreiecks*, p. 39 (1816). The property is probably much older.

(4) Weddle in the *Diary* for 1843, p. 75.

(10) The first values of l_1 l_2 l_3 are given by Vecten in Gergonne's *Annales*, IX., 304 (1818-9); the second values by Matthes in his *Commentatio*, p. 42 (1831).

(11) The first values are given by Weddle in the *Diary* for 1848, p. 78; the second values by Matthes in his *Commentatio*, p. 58 (1831).

(12) Mr Robert E. Anderson.

(14) The first equality is given by Mr Launoy in Bourget's *Journal de Mathématiques Élémentaires*, III. 160 (1879).

(15) The first equality is given in J. A. Grunert's article "Dreieck" in *Supplemente zu Klügel's Wörterbuche der reinen Mathematik*, I. 709 (1833). In this article Grunert gives also (20).

(16)—(19) Mr Robert E. Anderson.

(21) First part in Jacobi's *De Triangulorum...Proprietatibus*, p. 8 (1825). Both parts certainly much older.

(22)—(24) Jacobi, p. 13 (1825).

(25) Jacobi, p. 12 (1825), gives the first equality in the first alternative form.

(26) First equality given by Matthes in his *Commentatio*, p. 42 (1831).

(27) First equality given by Marsano in his *Considerazioni sul Triangolo Rettilineo*, p. 29 (1863).

(28) The value of $l_1 l_2 l_3$ is given by Vecten in Gergonne's *Annales*, IX. 304 (1819); that of $\lambda_1\lambda_2\lambda_3$ by Weddle in the *Diary* for 1848, p. 78.

(29) The first value is given by Vecten in Gergonne's *Annales*, IX. 305 (1819).

(31), (32) J. W. Elliott in the *Diary* for 1851, p. 58.

(33) The first of these eight values is given by Jacobi, p. 10 (1825).

(35) Weddle in the *Diary* for 1848, p. 78.

(36) ,, ,, ,, ,, ,, ,, p. 81.

(37) ,, ,, ,, ,, ,, ,, p. 80.

(38) *Nouvelles Annales*, 2nd series, IX. 548 (1870).

(39)—(42) Weddle in the *Diary* for 1848, pp. 81-2.

(43)—(45) Matthes, pp. 46, 48, 50 (1831), gives several of these proportions, but they must all have been known long previously.

(46) Matthes, p. 58 (1831), gives the values of λ_1 λ_2 λ_3, but he does not seem to have observed the corresponding ones for l_1 l_2 l_3.

(47) Matthes, pp. 48, 51, gives the first half of these values, the first two of (48), and the first of (49).

(50) The first two proportions are given by Jacobi, pp. 11, 19 (1825); the last two by Matthes, pp. 48, 51 (1831).

(51) The last proportion is given by Matthes, p. 51.

(52) J. W. Elliott in the *Diary* for 1851, p. 58. The equality of the last two expressions is given by Vecten in Gergonne's *Annales*, IX. 305 (1819).

(53)—(58) Weddle in the *Diary* for 1848, pp. 76-78, 82. The values in (58) are probably much older than this.

(61) Weddle in the *Diary* for 1848, p. 82.

(62) Value of AU^2 is given by William Mawson in the *Diary* for 1845, p. 67.

(63) Adams's *Eigenschaften des...Dreiecks*, p. 75 (1846).

(65)—(68) Weddle in the *Diary* for 1848, p. 83. The first equality in (67) is given by Adams in his *Eigenschaften des...Dreiecks*, p. 61 (1846).

(69), (71), (72), (74), (75) Matthes, pp. 47, 44, 59 (1831).

Formulae connected with the Radii of the Incircle and the Excircles of a Triangle.

By J. S. MACKAY, M.A., LL.D.

The following formulae may be added to the list given in the *Proceedings of the Edinburgh Mathematical Society*, Vol. XII., pp. 86–102 (1894).

$$\left.\begin{aligned} a &= \frac{r_1(r_2+r_3)}{s} = \frac{r(r_2+r_3)}{s_1} \\ &= \frac{s^2(r+r_3)+r_3^2(r_1-r_2)}{2sr_3} = \frac{s_1^2(r_1-r_2)+r_2^2(r+r_3)}{2s_1r_2} \end{aligned}\right\} \qquad (81)$$

and so on.

$$b+c = \frac{s(r_1+r)}{r_1} = \frac{s_1(r_1+r)}{r}, \;\ldots\ldots \qquad (82)$$

$$b-c = \frac{r_1(r_2-r_3)}{s} = \frac{r(r_2-r_3)}{s_1}, \;\ldots\ldots \qquad (83)$$

$$\left.\begin{aligned} (b+c)(c+a)(a+b) : abc &= h_1+h_2+h_3-r : r \\ (b+c)(a-c)(a-b) : abc &= h_1-h_2-h_3+r_1 : r_1 \end{aligned}\right\} \qquad (84)$$

$$\left.\begin{aligned} & 4r^2(r_1^2+r_2^2+r_3^2)+4(r_2^2r_3^2+r_3^2r_1^2+r_1^2r_2^2) \\ &= 4\,\Delta^4\left(\frac{1}{r_3^2r_3^2}+\frac{1}{r_3^2r_1^2}+\frac{1}{r_1^2r_2^2}+\frac{1}{r^2r_1^2}+\frac{1}{r^2r_2^2}+\frac{1}{r^2r_3^2}\right) \\ &= (a^2+b^2+c^2)^2-8\Delta^2 \end{aligned}\right\} \qquad (85)$$

$$\Delta^4\left(\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right)^2 = (a^2+b^2+c^2)^2 \qquad (86)$$

$$2\Delta^4\left(\frac{1}{r^4}+\frac{1^{\,4}}{r_1}+\frac{1}{r_2^4}+\frac{1}{r_3^4}\right) = (a^2+b^2+c^2)^2+8\Delta^2 \qquad (87)$$

$$\left.\begin{aligned} r : r_1 &= h_1 - 2r : h_1 = h_1 : h_1 + 2r_1 \\ r : r_2 &= h_2 - 2r : h_2 = h_2 : h_2 + 2r_2 \\ r : r_3 &= h_3 - 2r : h_3 = h_3 : h_3 + 2r_3 \end{aligned}\right\} \quad (88)$$

$$\left.\begin{aligned} r^2 : r_1^2 &= h_1 - 2r : h_1 + 2r_1 \\ r^2 : r_2^2 &= h_2 - 2r : h_2 + 2r_2 \\ r^2 : r_3^2 &= h_3 - 2r : h_3 + 2r_3 \end{aligned}\right\} \quad (89)$$

$$\frac{(h_1 - 2r)(h_2 - 2r)(h_3 - 2r)}{(h_1 + 2r_1)(h_2 + 2r_2)(h_3 + 2r_3)} = \frac{r^4}{s^4} \quad (90)$$

On the real Common Chords of a Point Circle and Ellipse.

By Robert Frederick Davis, M.A.

(1) If O be a given point in the plane of a given conic, the mutual relationship between point and conic is marked, first and foremost, by the existence of a certain determinate straight line (which is always real) known as the polar of O with respect to the conic. Next following the polar in natural order of sequence, come a certain pair of determinate straight lines:—

The single real pair of common chords of the conic and a point circle at O.

(2) As in the generality of cases, it will be best to rely upon analysis for discovery of facts, and then to look to geometry for elucidation.

(3) Consider the conic represented by the equation

$$x^2+y^2=(ax+by+c)(a'x+b'y+c') \quad \ldots \quad \ldots \quad \text{(A)}$$

which really involves five constants, for either c or c' may be put $=1$. It represents the locus of a point which moves in such a manner that the square of its distance from the origin varies as the product of its perpendicular distances upon the fixed straight lines

$$\left.\begin{aligned} ax+by+c&=0\\ a'x+b'y+c'&=0\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \text{(B)}$$

(4) These two straight lines (B) must be regarded as the real common chords of the conic and a point circle at O. They cannot intersect the conic in real points, and consequently lie outside the conic. Since any circle theoretically intersects a conic in four points lying two and two upon three pairs of common chords, and (A) may be written in the form

$$y^2-\iota^2x^2=(ax+by+c)(a'x+b'y+c'), \quad (\iota^2=-1)$$

therefore $y \pm ix$ is one pair of imaginary chords; and of the other two pairs, one pair must be real and one imaginary.

(5) It may here be noted that the straight lines (B) are equally inclined to either axis of the conic following the general property of the chords of intersection of a circle and conic. For the coefficient of xy vanishes in (A) when $ab' + b'a = 0$; so that perhaps a more convenient form might therefore be taken as

$$x^2 + y^2 = (\lambda x + \mu y + \nu)(\lambda x - \mu y + \nu')$$

Again, when (B) represents coincident straight lines, the origin becomes a focus and the coincident common chords become the corresponding directrix.

(6) The geometrical connexion of the foregoing is as follows:— If the conic be reciprocated with respect to any point O we get a second conic. The foci of the first reciprocate into certain straight lines $\Delta\Delta'$, $\delta\delta'$; while the property that the product of the perpendiculars from the foci upon any tangent to the first conic is constant reciprocates into the property that the (distance OP)^2 of any point on the second conic from O varies as the product of the perpendiculars from P on $\Delta\Delta'$, $\delta\delta'$. (By using Salmon's Theorem.) Thus from Art. (3) we see that $\Delta\Delta'$, $\delta\delta$ are the real common chords of intersection of a point circle at O with the second conic.

(7) It will be convenient to designate the above pair of straight lines the Delta lines of the conic corresponding to O.

(8) Properties of the Delta lines are therefore easily found by reciprocating focal properties; as, for instance, the property that the tangent at any point is equally inclined to the focal distances becomes by reciprocation the part of any tangent intercepted by the Delta lines is divided at the point of contact into two segments which subtend equal or supplementary angles at O.

(9) The Delta lines intersect upon the polar of O. Also if Σ, σ be the poles of the Delta lines, then CΣ, Cσ are equally inclined to either axis. (The points Σ, σ correspond to the directrices of the reciprocated conic.)

FIGURE 25.

(10) Let UU′ be a straight line outside an ellipse meeting the ellipse in the imaginary points ω, ω'. Then T the middle point of

$\omega\omega'$ is real, and is found by drawing the tangent at P parallel to UU' and producing CP to meet UU' in T.

Also ωT (or $=\omega'$T) is given by the equation

$$\frac{CT^2}{CP^2}+\frac{\omega T^2}{CD^2}=1$$

where since CT>CP ωT^2 is negative. Through T draw TO $\perp^r$ to UU' (*i.e.*, parallel to the normal at P). Then if a point circle at O pass through ω, ω'

$$OT^2+\omega T^2=0,$$

hence
$$\frac{CT^2}{CP^2}-\frac{OT^2}{CD^2}=1.$$

Hence O lies on the concentric hyperbola passing through P whose conjugate diameter is $=$CD and is perpendicular to CD. This is obviously the confocal hyperbola through P (for at their point of intersection CP is a common semi-diameter, and the conjugate semi-diameter $=(\text{SP}\,.\,\text{SP})^{\frac{1}{2}}$ in each case though in perpendicular directions).

(11) Since in Fig. 25 O may lie on either side of UU', we see that to any line outside an ellipse correspond two determinate points the point-circles at which have the given line as their common chord with the ellipse.

(12) To determine therefore the Delta lines corresponding to any point O with respect to a given ellipse, we have the following construction. Draw the confocal hyperbola through O intersecting the ellipse at the extremities of the equi-diameters PCP', pCp'; draw OT parallel to the normal at P to meet CP produced in T; and through T draw UU' perpendicular to OT or parallel to the tangent at P. Then UU' is one of the Delta lines required: and a similar construction gives the other.

(13) If upon CP, Cp points Q, q be taken respectively such that $\text{CQ}\,.\,\text{CT}=\text{CP}^2=\text{C}p^2=\text{C}q\,.\,\text{C}t$, then Q, q are obviously the points in which the tangent at O to the confocal hyperbola meets CP, Cp respectively. Consequently Q, O, q are in a straight line; and UU', uu' intersect on the polar of O. See Art. (9).

(14) Various geometrical properties may be noted. By reciprocation we find that if R be any point on $\Delta\Delta'$ (see Fig. 26*a*), and $R\Sigma$ intersect the curve in Q, Q′, then OQ, OQ′ each divide the angle ΣOR into parts whose sines are in the ratio $e:1$, and consequently the range $\{RQ\Sigma Q'\}$ is harmonic, as we should expect. This interesting result may be otherwise stated: If BC be the fixed base of a triangle whose variable vertex P describes a straight line, the locus of Q taken on PC such that $\sin QBC : \sin QBP$ is constant is a conic.

(15) Again (in Fig. 26*b*), if the tangent at Q meet $\Delta\Delta'$ in Z, and $Q\Sigma$ meet $\Delta\Delta'$ in R (and the curve in Q′), ZOR is a right angle. Whence ZQ′ is the tangent at Q′; and the conjugate points Z, R subtend a right angle at O. Thus given a straight line outside an ellipse, the circle described upon any pair of conjugate points lying on that straight line passes through two fixed points—the point circles at which have the given straight line as their chord of intersection with the ellipse.

(16) The pedal of an ellipse with respect to O is found by reciprocation to be the locus of a point whose distances from three points (one of which is O and the others the feet of the perpendiculars from O on the Delta lines) are connected by a linear relation (bicircular quartic).

(17) Since in Fig. 26*b*, OQ^2 varies as the product of perpendiculars from Q on Delta lines, OQ is a maximum (therefore OQ normal at Q) when this product is constant for two consecutive positions of Q. From the properties of the rectangular hyperbola we know that this is the case when the tangent at Q intercepted between the Delta lines is bisected at the point of contact. Thus we are led to the conclusion that for any pair of Delta lines the tangent to the ellipse can assume four positions in which the intercepted part is bisected by the point of contact, and consequently the points Z, Z′ are then equi-distant from O.

(18) I have already shown in the Reprint to the *Educational Times*, Vol. LIV., Appendix 1, that when $\Delta\Delta'$ touches the ellipse and consequently Σ lies on the curve, σ in this case becomes the Frégier point of Σ and $\delta\delta'$ the Frégier line.

(19) To the best of my belief the late Professor Wolstenholme, in his monumental collection of problems (containing as it does such a vast store of interesting matter relating to Conics), does not indicate any construction for the Delta lines, or specify their equations.

Dr. James Booth was evidently aware of their existence, and, in the special case only in which O lies on the axis, employs their properties in his "New Geometrical Methods," Vol. I

Dr. Taylor, in his "Geometry of Conics," mentions their existence, but gives no construction.

I conclude with the following analytical notes.

I. To find the value of λ for which the equation

$$(x-\alpha)^2+(y-\beta)^2-\lambda\left(\frac{x^3}{a^2}+\frac{y^2}{b^2}-1\right)=0 \quad \ldots \quad \ldots \quad \text{(A)}$$

represents straight lines; in other words, to find the common chords of the ellipse and a point-circle at α, β.

The discriminant must $=0$, or

$$\left(1-\frac{\lambda}{a^2}\right)\left(1-\frac{\lambda}{b^2}\right)(\alpha^2+\beta^2+\lambda)-\left(1-\frac{\lambda}{a^2}\right)\beta^2-\left(1-\frac{\lambda}{b^2}\right)\alpha^2=0.$$

It will be found that one value of $\lambda=0$, as we should *a priori* expect since the lines $(y-\beta)=\pm i(x-d)$ form one pair of common chords.

The other two values of λ are given by the equation

$$\lambda^2+\lambda(\alpha^2+\beta^2-a^2-b^2)-(b^2\alpha^2+a^2\beta^2-a^2b^2)=0. \quad \ldots \quad \text{(B)}$$

Note if α, β lies on the ellipse, another value of $\lambda=0$, and the third value of $\lambda=a^2+b^2-\alpha^2-\beta^2$,
and the real common chords are

$$(x-\alpha^2)+(y-\beta)^2=(a^2+b^2-\alpha^2-\beta^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

which may be thrown into the form

$$(a^2-b^2)\left\{\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}-1\right\}\left\{\frac{x\alpha}{a^2}-\frac{y\beta}{b^2}-\frac{a^2+b^2}{a^2-b^2}\right\}=0$$

the factors representing the tangent at (α, β) and the Frégier line corresponding to α, β.

Again, if α, β lies on the orthocyclic circle the two values of λ are $= \pm(b^2\alpha^2 + a^2\beta^2 - a^2b^2)$

From (A) the lines will be real or not, according as the parallels through the origin are real or not: that is, according as

$$\left(1 - \frac{\lambda}{a^2}\right)x^2 + \left(1 - \frac{\lambda}{b^2}\right)y^2 = 0$$

has real or imaginary roots.

Since this may be written

$$y^2 = \frac{a^2 - \lambda}{\lambda - b^2} \cdot \frac{b^2}{a^2} x^2$$

we see that λ must lie between b^2 and a^2.

Now in (B) if we substitute $\lambda = a^2$ and $\lambda = b^2$ we get $(a^2 - b^2)\alpha^2$ and $(b^2 - a^2)\beta^2$ respectively, the first of which is positive and the second negative. This shows that (B) has always a real root between b^2 and a^2.

II. If P be a point on the ellipse whose excentric angle is α, the coordinates of P are $a\cos\alpha$, $b\sin\alpha$. It can be easily verified that the semi-axes of the confocal hyperbola through P are

$$c\cos\alpha,\ c\sin\alpha \quad \text{where} \quad c^2 = a^2 - b^2.$$

For $$\frac{a^2\cos^2\alpha}{c^2\cos^2\alpha} - \frac{b^2\sin^2\alpha}{c^2\sin^2} = 1, \quad \text{and} \quad c^2\cos^2\alpha + c^2\sin^2\alpha = c^2.$$

If O be any point on this confocal hyberbola its coordinates may be taken as $c\cos\alpha\sec\phi$ and $c\sin\alpha\tan\phi$.

Now I have actually verified that the equations

$$(x - c\cos\alpha\sec\phi)^2 + (y - c\sin\alpha\tan\phi)^2$$
$$= \left\{\frac{c}{a}x\cos\alpha + \frac{c}{b}y\sin\alpha - (a\sec\phi - b\tan\phi)\right\}$$
$$\times \left\{\frac{c}{a}x\cos\alpha - \frac{c}{b}y\sin\alpha - (a\sec\phi + b\tan\phi)\right\}$$

and $$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)(a^2\sin^2\alpha + b^2\cos^2\alpha) = 0$$

are identically equal.

This shows that the real chords of the point-circle at the point whose coordinates are $c\cos\alpha\sec\phi$ $c\sin\alpha\tan\phi$ are the straight lines

$$\left(\frac{c}{a}x\cos\alpha - a\sec\phi\right) \pm \left(\frac{c}{b}y\sin\alpha + b\tan\phi\right),$$

which expressions are probably new.

Note on Triangle Transformations.

By R. F. Muirhead, M.A.

The investigation given in the following Note was suggested by a passage in the paper by Mr Lemoine, presented to the Society by Dr. Mackay at a recent meeting. The main subject of that paper is what he terms the "*Transformation continue dans le triangle et dans le tétraèdre*"; for the explanation of that phrase and other terms connected with it, the reader is referred to the paper just mentioned. In the notation for the quantities connected with the triangle, however, I shall follow Dr. Mackay's system as explained in his paper in Vol. I. of our *Proceedings*.

I shall use, moreover, the letters α, β, γ provisionally as symbols of operation, to denote what Mr Lemoine calls "*la transformation continue en* A, *en* B, *en* C" respectively. Thus, as shown in Lemoine's paper,

$$\alpha a = a, \ \ \alpha b = -b, \ \ \alpha \mathrm{A} = -\mathrm{A}, \ \ \alpha \mathrm{B} = \pi - \mathrm{B}, \text{ etc.}$$

A compound operation such as $\beta\alpha . \mathrm{A}$ I shall use to mean $\beta(\alpha \mathrm{A})$, and $\dfrac{1}{\alpha}$ to denote the reverse of the operation α. Thus $\alpha\mathrm{B} = \pi - \mathrm{B}$, and $\mathrm{B} = \dfrac{1}{\alpha}(\pi - \mathrm{B})$ are taken as equivalent equations.

When an equation like $\alpha = \beta$ occurs, it shall mean that, with respect to the functions considered, the operation α is equivalent to the operation β; and $\alpha = 1$ shall mean that the operation α leaves the functions under consideration unchanged.

Now in Mr Lemoine's paper he gives a list of four kinds of cases, with respect to the variety of results got by applying α, β and γ to triangle-identities, which may be indicated by the following typical cases:

(1) $\alpha = \beta = \gamma = 1$
(2) $\alpha, \beta, \gamma, 1$ all different
(3) $\alpha = 1 \neq \beta = \gamma$
(4) $\alpha = \beta = \gamma \neq 1$

He states that all these are found to occur, but that he has not yet found any case of the following type:

$$(5) \quad \alpha = 1, \quad \beta, \gamma, 1 \quad \text{all different.}$$

The object of this Note is to account for the non-occurrence of such cases. We may complete the list of typical cases by

$$(6) \quad \alpha \neq 1, \quad \beta = \gamma = 1$$
$$(7) \quad \alpha \neq 1, \quad \beta = \gamma \neq 1.$$

We have
$$\gamma(a) = -a$$
$$\beta\gamma(a) = \beta(-a) = a$$
$$\alpha\beta\gamma(a) = \alpha(a) = a$$

Similarly $\qquad \alpha\beta\gamma(r) = -r, \quad \alpha\beta\gamma(r_1) = -r_1, \quad \text{etc.}$

In fact, if F denote any function whatever, we have

$$\left.\begin{aligned} &\alpha\beta\gamma\{F(a, b, c, s, s_1, s_2, s_3, r, r_1, r_2, r_3, h_1, h_2, h_3, \triangle, R)\} \\ &= \{F(a, b, c, s, s_1, s_2, s_3, -r, -r_1, -r_2, -r_3, \\ &\qquad\qquad -h_1, -h_2, -h_3, -\triangle, -R)\} \end{aligned}\right\}$$

Thus the compound operator $\alpha\beta\gamma$ changes the signs of certain letters, but leaves them otherwise unaltered.

Now if any identity connecting the functions of the general triangle be written in the form $F = 0$, it is clear that the identity will not be altered by applying the operation $\alpha\beta\gamma$, so long as F consists of terms which, with respect to the letters $r, r_1, \cdots\cdots R$ (whose signs are changed thereby) are either all of *even* dimensions, or all of *odd* dimensions. Such terms will hereafter be referred to, for brevity's sake, as *even*, or *odd* simply. The letters $ab \cdots\cdots s_3$ whose signs are unchanged by the operation $\alpha\beta\gamma$ are not to be reckoned in this connection. Omitting for the present, then, all reference to identities in which F is a *mixed* function, we may say that $\alpha\beta\gamma = 1$, *i.e.* the operator $\alpha\beta\gamma$ reproduces the same identity as we start with.

Now it is easily proved that $\alpha^2 F = F$, whatever the function F may be, or, as we may write it $\alpha^2 = 1$, whence $\alpha = \dfrac{1}{\alpha}$. And the same is true as applied to an identical equation.

But since $\alpha\beta\gamma = 1$, restricting ourselves to identities which are not *mixed*, we have $\alpha . \alpha\beta\gamma = \alpha$

$$\therefore \quad \alpha^2\beta\gamma = \alpha \qquad\qquad \therefore \quad \beta\gamma = \alpha$$

Hence if $\alpha = 1$, $\beta\gamma = 1$ $\qquad \therefore \quad \beta = \dfrac{1}{\gamma} = \gamma$

This shows that case (5) cannot occur.

Conversely if $\beta = \gamma$, we have $\beta\gamma = \gamma^2 = 1$ $\qquad \therefore \quad \alpha = 1$

This shows that cases (6) and (7) cannot occur.

We may sum up these results by saying that if $\alpha = 1$, then $\beta = \gamma$; and *vice versa*.

Now it may be asked, Are there not *mixed* identities which give results of the types (5), (6), (7)? The answer is that there are; but they are always composite identities, which may be reduced to two or more simpler ones. In fact, if any identity be expressed by $F = 0$ when $F \equiv F_1 + F_2$, F_1 being an *odd*, and F_2 an *even* function, as explained above, then applying $\alpha\beta\gamma$ to the identity $F_1 + F_2 = 0$ we get a new identity $-F_1 + F_2 = 0$ whence $F_1 = 0$ and $F_2 = 0$ identically; *i.e.* $F = 0$ is composed by adding together two identities $F_1 = 0$ and $F_2 = 0$. It is easy to manufacture such cases, *e.g.*,

$$sr - s_1r_1 + rr_1 - s_2s_3 = 0.$$

This is an identity which belongs to the type (5).

I have in the foregoing omitted all consideration of the angles A, B, C, for the reason that they introduce further complication. In fact

$$\alpha\beta\gamma F(A, B, C) = F(-A, -B + 2\pi, -C)$$

so that the characteristic property of $(\alpha\beta\gamma)$ is less simple. It is clear, however, that so long as only such functions of A, B, C as are unaltered except in sign by substituting $-A + 2\pi$ for $-A$, etc., the results will be the same as before, A, B and C being reckoned along with r, r_1 etc., in counting dimensions.

Thus any formula may contain any trigonometrical function of A or of any multiple of A, also $\cot\dfrac{A}{2}$, $\tan\dfrac{A}{2}$, without making any difference in our conclusions But if such functions as $\cos\dfrac{A}{2}$, $\sin\dfrac{A}{3}$ occur, the above conclusions cannot be drawn; and I doubt whether in such cases we could even depend on the validity of the "transformation continue" itself, without special precautions.

Fourth Meeting, February 8th, 1895.

J. S. MACKAY, Esq., M.A., LL.D., F.R.S.E., in the Chair.

Theorems in the Products of Related Quantities.

By F. H. JACKSON, M.A.

§ 1. The object of this Note is to prove the following theorems:

$$\left.\begin{aligned}(a+b)_{-n} = a_{-n} - na_{-n-1}b_1 + \frac{n\cdot n+1}{2\,!}a_{-n-2}b_2 \\ - \frac{n\cdot n+1\cdot n+2}{3\,!}a_{-n-3}b_3 + \cdots\end{aligned}\right\} \quad (1)$$

in which $a_{-n} = \dfrac{1}{a+n\cdot a+n-1\cdots\cdot a+1\cdot}$ and the series is subject to conditions for convergence.

$$\left.\begin{aligned}\frac{P(0)}{x+n} - {}_nC_1\frac{P(y)}{x+n-1} + {}_nC_2\frac{P(2y)}{x+n-2} - \cdots \text{ to } n+1 \text{ terms} \\ \equiv (-1)^n\frac{x\,!\,n\,!}{x+n\,!}\cdot\frac{P(xy+ny)}{x}\end{aligned}\right\} \quad (2)$$

$$P(0) - {}_nC_1P(y) + {}_nC_2P(2y) - \cdots \text{ to } n+1 \text{ terms} \quad \equiv 0 \qquad (3)$$

$$N(0) - {}_nC_1N(y) + {}_nC_2N(2y) - \cdots \quad ,, \quad ,, \quad ,, \quad \equiv (-y)^n.\,n\,! \quad (4)$$

in which $P(y) \equiv (a+y)(b+y)(c+y) \cdots$ to p factors

$N(y) \equiv (a+y)(b+y)(c+y) \ldots$ to n factors

and the quantities $abcd\cdots x$ are unrestricted. In theorem (2) $p \lesseqgtr n$, but in (3) $p<n$

The following theorems are derived from the above

$$\left.\begin{aligned}\frac{(a)_p}{x} - {}_nC_1\frac{(a-1)_p}{x-1} + {}_nC_2\frac{(a-2)_p}{x+2} - \cdots\cdots\cdots + (-1)^n\frac{(a-n)_p}{x+n} \\ \equiv \frac{x\,!\,n\,!}{x+n\,!}\;\frac{(x+a)_p}{x}\end{aligned}\right\} \quad (5)$$

$$\frac{(a)_p}{x} - {}_nC_1\frac{(a+1)_p}{x-1} + \cdots\cdots\cdots\cdots \equiv (-1)^n\frac{x-n\,!\,n\,!}{x\,!}\;\frac{(x+a)_p}{x-n} \qquad (6)$$

$$(p \lesseqgtr n)$$

$$(x)_n - {}_nC_1(x+y)_n + \cdots \quad + (-1)^n(x+ny)_n \equiv (-y)^n n\,! \qquad (7)$$

$$(x)_p - {}_nC_1(x+y)_p + \cdots + (-1)^n(x+ny)_p \equiv \quad 0 \qquad (p<n) \qquad (8)$$

These correspond in form with the following theorems in the products of equal quantities (powers)

$$\frac{(a)^p}{x} - {}_nC_1\frac{(a-1)^p}{x+1} + \cdots\cdots + (-1)_n\frac{(a-n)^p}{x+n} \equiv \frac{x\,!\,n\,!}{x+n\,!}\;\frac{(x+a)^p}{x} \qquad (9)$$

and three others formed from (6) (7) and (8) by changing subscript letters to indices, thus $(x+y)_n$ to $(x+y)^n$.

§2. The theorems in powers corresponding to (7) and (8) are given on page 372 of C. Smith's "Treatise on Algebra," and are mentioned here because of their similarity to (7) and (8). All the other identities are new to me. It would appear that, from most of the algebraical identities involving positive integral powers, other identities may be derived by substituting suffixes for indices. Thus Vandermonde's Theorem corresponds to the Binomial Theorem, and it is easily deduced that

$$(a+b+c+\cdots \text{ to } m \text{ terms})_n \equiv \Sigma\left\{\frac{(a)_p(b)_q(c)_r\cdots}{p\,!\,q\,!\,r\,!\cdots} \text{to } m \text{ factors}\right\},$$

$pqrs\cdots\cdot n$ being positive integers subject to $p+q+r+s+\cdots=n$. This corresponds to the Multinomial Theorem. It is possible to extend this for negative values of n, a_{-n} being interpreted as in theorem (1). Both the theorems (5) and (9) are particular cases of (2). May not the Binomial Theorem and Vandermonde's Theorem be special cases of some general theorem?

Vandermonde's Theorem.

§ 3. Can any meaning be attached to the theorem

$$(a+b)_n = a_n + n_1 a_{n-1} b_1 + \frac{n_2}{2\,!} a_{n-2} b_2 + \cdots$$

when n is not restricted, as hitherto, to being a positive integer? In Vandermonde's Theorem a_n represents the product of n related factors $a \cdot a-1 \cdot a-2 \cdots\cdots a-n+1$, and certainly so long as we regard a_n as the product of n factors such expressions as

$$a_{-n}\,, \quad a_{\frac{1}{2}}\,, \quad a_{\frac{p}{q}}\,,$$

seem beyond our comprehension. Exactly the same might have been written of the quantities

$$a^{-n}, \quad a^{\frac{p}{q}}, \quad a^{\frac{1}{2}},$$

so long as a^m was regarded as the product of m factors each equal to a. The Binomial Theorem, until fractional and negative indices were interpreted, was a finite algebraical identity; but as soon as a fundamental law $a^m \times a^n = a^{m+n}$, was assumed in the Theory of Indices, then the expressions

$$a^{-m}, \quad a^{\frac{p}{q}},$$

were interpreted, and the Binomial Theorem was shown to hold (with certain restrictions) for fractional and negative powers.

§ 4. Now in the expressions a_m, a_n, (the usual meanings being attached) we have these relations

$$a_n \times (a-n)_m \quad = a_{m+n} \qquad (\alpha)$$

$$a_m \times (a-m)_n \quad = a_{m+n} \qquad (\beta)$$

$$a_n \times (a-n)_{m-n} = a_m \qquad (\gamma)$$

These are all expressions of one law. Let us assume this law as general and interpret a_{-n}, a_o, in accordance with our assumption.

In the relation (α) put $n=0$ then we have $a_o \times a_m = a_m$ whence $a_o = 1$ and this is analogous to $a^o = 1$.

In the relation (γ) change n to $-r$ then we obtain
$$a_{-r} \times (a+r)_{m+r} = a_m$$
$$\therefore \quad a_{-r} = \frac{a_m}{(a+r)_{m+r}}$$

If m and r be integers $a_{-r} = \dfrac{1}{(a+r)(a+r-1)\ldots(a+1)}$ which is analogous to $a^{-r} = \dfrac{1}{a \cdot a \cdot a \cdots}$ to r factors.

From the relations (α) and (β) we get
$$a_n \times (a-n)_m = a_m \times (a-m)_n$$

make $n = \dfrac{p}{q}$ then we have $\dfrac{a_{\frac{p}{q}}}{(a-m)_{\frac{p}{q}}} = \dfrac{a_m}{\left(a - \frac{p}{q}\right)_m}$

Supposing that m is a positive integer, this equation gives the ratio of any two functions $a_{\frac{p}{q}}$, $b_{\frac{p}{q}}$ in which a and b differ by an integer m, viz.:
$$a_{\frac{p}{q}} = b_{\frac{p}{q}} \cdot \frac{a \cdot a-1 \cdot a-2 \cdots a-m+1}{a-\frac{p}{q} \,.\, a-\frac{p}{q}-1 \ldots \; a-\frac{p}{q}-m+1}$$

and again when a is an integer
$$a_{\frac{p}{q}} = (0)_{\frac{p}{q}} \cdot \frac{a\,!}{\left(a-\frac{p}{q}\right)\left(a-\frac{p}{q}-1\right)\ldots\left(1-\frac{p}{q}\right)} .$$

The function $(a)_{\frac{p}{q}}$ will be discussed in another paper.

$a_{\frac{1}{n}}$ must be a function of a possessing the property
$$a_{\frac{1}{n}} \times \left(a-\frac{1}{n}\right)_{\frac{1}{n}} \times \left(a-\frac{2}{n}\right)_{\frac{1}{n}} \ldots \times \left(a-\frac{n-1}{n}\right)_{\frac{1}{n}} = a .$$

We now proceed to prove Theorem (1).

§ 5. Denote the infinite series (1) by $f(-n)$

thus $$f(-n) \equiv a_{-n} + (-n)_1 a_{-n-1} b_1 + \frac{(-n)_2}{2\,!} a_{-n-2} b_2 + \cdots \qquad \text{(A)}$$

Now $$\left.\begin{aligned}(a+b+n)_m = (a+n+r)_m + m_1(a+n+r)_{m-1}(b-r)_1 \\ + \frac{m_2}{2\,!}(a+n+r)_{m-2}(b-r)_2 + \cdots\cdots + (b-r)_m\end{aligned}\right\} \qquad \text{(B)}$$

m being a positive integer $> n$

Multiply $f(-n)$ by $(a+b+n)_m$ in the following manner:

a_{-n} by the series on the right side of B putting $r=0$

$(-n)_1 a_{-n-1} b_1$ ... ... $r=1$, etc.

Then we obtain

$$\left.\begin{aligned}
&f(-n) \times (a+b+n)_m \\
&\equiv a_{-n}\Big[(a+n)_m + m_1(a+n)_{m-1}b_1 + \frac{m_2}{2\,!}(a+n)_{m-2}b_2 + \ldots + b_m\Big] \\
&+ (-n)_1 a_{-n-1} b_1 \\
&\quad \times \Big[(a+n+1)_m + \quad \cdots \quad \cdots \quad \cdots \quad + (b-1)_m\Big] \\
&+ \frac{(-n)_2}{2\,!} a_{-n-2} b_2 \\
&\quad \times \Big[(a+n+2)_m + \quad \cdots \quad \cdots \quad \cdots \quad + (b-2)_m\Big]
\end{aligned}\right\} \qquad \text{(C)}$$

+ an infinite number of brackets similar to the above.

Now with the interpretation of symbols in §§ (4) and (1).

$$a_{-n-s} \times (a+n+s)_m = a_{m-n-s} \quad \text{and} \quad b_s \times (b-s)_r = b_r$$

∴ The expression (C) becomes

$$\begin{aligned}
&\Big[a_{m-n} + m_1 a_{m-n-1} b_1 + \frac{m_2}{2\,!} a_{m-n-2} b_2 + \ldots + a_{-n} b_m\Big] \\
&+ (-n)_1 \cdot \Big[a_{m-n-1} b_1 + m_1 a_{m-n-2} b_2 + \quad \cdots \quad + a_{-n-1} b_{m+1}\Big] \\
&+ \frac{(-n)_2}{2\,!}\Big[a_{m-n-2} b_2 + \quad \cdots \quad \cdots \quad \cdots \quad + a_{-n-2} b_{m+1}\Big]
\end{aligned}$$

+ etc. to infinity.

Collect the resulting terms diagonally.

We obtain a_{m-n}

$$+a_{m-n-1}b_1\left[m_1+(-n)_1\right]$$

$$+a_{m-n-2}b_2\left[\frac{m_2}{2\,!}+\frac{m_1(-n)_1}{1\,!\,1\,!}+\frac{(-n)_2}{2\,!}\right]$$

$$\cdots\cdots\cdots\cdots$$

$$+\quad a_0 b_{m-n}\left[\frac{m_{m-n}}{m-n\,!}+\cdots+\frac{(-r)_{m-n}}{m-n\,!}\right]$$

$+\cdots\cdot$ an infinite number of brackets similar to the above.

Now all brackets after the $m-n+1^{th}$ vanish identically by Vandermonde's Theorem since $m-n$ is a positive integer, and the expression becomes

$$a_{m-n}+(m-n)_1 a_{m-n-1}b_1+\frac{(m-n)_2}{2\,!}a_{m-n-2}b_2+\cdots+b_{m-n}\equiv(a+b)_{m-n}$$

$\therefore$ we have proved $(a+b)_{m-n}=(a+b+n)_m\times f(-n)$

whence $f(-n)=\dfrac{(a+b)_{m-n}}{(a+b+n)_m}=\dfrac{1}{(a+b+n)(a+b+n-1)\cdots(a+b+1)}$

which in our notation $=(a+b)_{-n}$

Conditions for the convergence of $f(-n)$ can easily be obtained.

§ 6. To prove Theorems (2), (3), (4), etc.

It is well known that

$$1-\frac{x}{x+1}\,{}_nC_1+\frac{x}{x+2}\,{}_nC_2-\text{to } n+1 \text{ terms}\equiv\frac{n!}{x+n\cdot x+n-1\cdots x+1\cdot}$$

we shall write the expression on the right as $\dfrac{x\,!\,n\,!}{x+n\,!}$ remembering that when x is not an integer it must be written in the long form. Multiply both sides of the identity by $x+a$ in the following way.

The first term by $x+a$

„ second „ „ $(x+1)+(a-1)$

„ $r+1^{th}$ „ „ $(x+r)+(a-r)$

we then obtain

$$x\left[1 - {}_nC_1 + {}_nC_2 - \cdots\right] + \left[a - \frac{x}{x+1}{}_nC_1(a-1) + \cdots\right] \equiv \frac{x\,!\,n\,!}{x+n\,!}(x+a)\,.$$

Now the first bracket on the left is identically equal to zero

$$\therefore \quad \frac{a}{x} - {}_nC_1\frac{a-1}{x+1} + {}_nC_2\frac{a-2}{x+2} - \cdots \text{ to } n+1 \text{ terms} \equiv \frac{x\,!\,n\,!}{x+n\,!}\,\frac{(x+a)}{x}$$

Proceeding in this way we shall finally obtain

$$\left[a^s - {}_nC_1(a-1)^s + \cdots\cdots + (-1)^n(a-n)^s\right]$$
$$+\left[\frac{a^{s+1}}{x} - {}_nC_1\frac{(a-1)^{s+1}}{x+1} + \cdots\cdots\right] \equiv \frac{x\,!\,n\,!}{x+n\,!}\,\frac{(x+a)^{s+1}}{x}$$

Now it is well known that the first bracket on the left $\equiv 0$ so long as s is an integer $< n$. $\therefore$ we have

$$\frac{a^p}{x} - {}_nC_1\frac{(a-1)^p}{x+1} + {}_nC_2\frac{(a-2)^p}{x+2} - \cdots\cdots \equiv \frac{x\,!\,n\,!}{x+n\,!}\,\frac{(x+a)^p}{x} \qquad \text{(D)}$$

p being an integer $\lesseqgtr n$.

This proves Theorem (9). Replacing $x+n$ by x and $a-n$ by a we obtain

$$\frac{a^p}{x} - {}_nC_1\frac{(a+1)^p}{x-1} + \cdots\cdots \text{ to } n+1 \text{ terms} \equiv \frac{x-n\,!\,n\,!}{x\,!}\cdot\frac{(x+a)^p}{x-n} \qquad \text{(E)}$$

§7. Take $S_1 = a+b+c+\cdots$ to p terms

$S_2 =$ the sum of all products of the letters, two at a time.

. . . .

$S_r =$ the sum of all products, r at a time.

Then

$$P(x+n) = S_p + (x+n)S_{p-1} + (x+n)^2S_{p-2} + \cdots + (x+n)^pS_o \qquad \text{(F)}$$

From Theorem (9)

$$(x+n)^p \equiv x \cdot \frac{x+n\,!}{x\,!\,n\,!}\left\{\frac{n^p}{x} - {}_nC_1\frac{(n-1)^p}{x+1} + \cdots\cdots + (-1)^n\frac{o^p}{x+n}\right\}$$

so long as $p \lessgtr n$. Substitute in (F) then

$$\mathrm{P}(x+n) \equiv \mathrm{S}_p\Big[n^\circ - \frac{x}{x+1}{}_n\mathrm{C}_1(n-1)^\circ + \frac{x}{x+2}{}_n\mathrm{C}_2(n-2)^\circ - \cdots\cdots\cdots$$

$$\cdots\cdots\cdots + (-1)^n\frac{x}{x+n}0^\circ\Big]\frac{x+n\,!}{x\,!\,n\,!}$$

$$+\mathrm{S}_{p-1}\Big[n - \frac{x}{x+1}{}_n\mathrm{C}_1(n-1) + \cdots\cdots\cdots\cdots$$

$$\cdots\cdots + (-1)^n\frac{x}{x+n}\cdot 0\Big]\frac{x+n\,!}{x\,!\,n\,!}$$

$$+\mathrm{S}_{p-2}\Big[n^2 - \frac{x}{x+1}{}_n\mathrm{C}_1(n-1)^2 + \cdots\cdots\cdots\cdots$$

$$\cdots\cdots + (-1)^n\frac{x}{x+n}\cdot 0^2\Big]\frac{x+n\,!}{x\,!\,n\,!}$$

$$\cdots\cdots\cdots\cdots$$

$$+\ \mathrm{S}_0\Big[n^p - \frac{x}{x+1}{}_n\mathrm{C}_1(n-1)^p + \cdots\cdots\cdots\cdots$$

$$\cdots\cdots + (-1)^n\frac{x}{x+n}0^p\Big]\frac{x+n\,!}{x\,!\,n\,!}$$

This expression may be written

$$\frac{x+n\,!}{x\,!\,n\,!}\bigg\{\Big[\mathrm{S}_p + n\mathrm{S}_{p-1} + n^2\mathrm{S}_{p-2} + \cdots\cdots + n^p\mathrm{S}_o\Big]$$

$$+\Big[\mathrm{S}_p + (n-1)\mathrm{S}_{p-1} + (n-1)^2\mathrm{S}_{p-2} + \cdots\cdots\Big]\frac{x+1}{x}{}_n\mathrm{C}_1$$

$$+\Big[\mathrm{S}_p + (n-2)\mathrm{S}_{p-1} + (n-2)^2\mathrm{S}_{p-2} + \cdots\cdots\Big]\frac{x+2}{x}{}_n\mathrm{C}_1$$

$$+\Big[\cdots\cdots\Big] + \cdots\cdots\cdots\cdots\bigg\}$$

$$\equiv \frac{x+n\,!}{x\,!\,n\,!}\left\{\mathrm{P}(n) - n\mathrm{C}_1\frac{x}{x+1}\mathrm{P}(n-1) + \cdots\cdots \text{ to } n+1 \text{ terms}\right\}$$

whence $\displaystyle \frac{x\,!\,n\,!}{x+n\,!}\mathrm{P}(x+n) \equiv \frac{\mathrm{P}(0)}{x+n} - n\mathrm{C}_1\frac{\mathrm{P}(1)}{x+n-1} + \cdots + (-1)^n\frac{\mathrm{P}(n)}{x}$

If we replace a. by $\frac{a}{y}$, b by $\frac{b}{y}$, etc., then

$$P(r) = \frac{(a+ry)(b+ry)\cdots\cdot}{y^p} = \frac{P(ry)}{y^p}$$

and we obtain the theorem in form (2), since y^p is common to all denominators and so divides out.

Replacing $x+n$ by r, we obtain

$$\frac{P(0)}{r} - nC_1\frac{P(y)}{r-1} + nC_2\frac{P(2y)}{r-2} - \cdots \equiv (-1)^n\frac{r-n\,!\,n\,!}{r\,!}\;\frac{P(ry)}{r-n} \quad \text{(G)}$$

§8. Again in Theorem (G) make

$$y=1, \quad r=x, \quad b=a-1, \quad c=a-2, \quad \text{etc.,}$$

then

$$P(ry) \equiv (a+r)(a+r-1)\cdots\cdot(a+r-p+1) \equiv (a+r)_p \equiv (a+x)_p$$

and we have

$$\frac{(a)_p}{x} - nC_1\frac{(a+1)_p}{x+1} + \text{ etc. } \cdots\cdot \equiv (-1)^n\frac{x-n\,!\,n\,!}{x\,!}\;\frac{(x+a)_p}{x-n}$$

This is the Identity (6).

If we replace $a+n$ by a, $b+n$ by $a-1$, $c+n$ by $a-2$, etc., we obtain

$$\frac{(a)_p}{x} - nC_1\frac{(a-1)_p}{x+1} + nC_2\frac{(a+2)_p}{x+2} - \cdots\cdot \equiv \frac{x\,!\,n\,!}{x+n\,!}\;\frac{(x+a)_p}{x}$$

This is the Identity (5).

The left side of (3) may be written

$$[S_p + 0\cdot S_{p-1} + 0^2S_{p-2} + \cdots\cdots + 0^pS_o]$$
$$-nC_1[S_p + yS_{p-1} + y^2S_{p-2} + \cdots\cdots + y_pS_o]$$
$$+(-1)^n nC_n[S_p + nyS_{p-1} + n^2y^2S_{p-2} + \cdots + n^py^pS_o]$$
$$\equiv S_p[1 - nC_1 + nC_2 - \cdots] + S_{p-1}[0 - nC_1\cdot y + nC_2 \,.\, 2y - \cdots] + \cdots\cdots$$
$$\cdots\cdots + S_o[0^p - nC_1y^p + nC_2 2^py^p - \cdots]$$

when p is an integer $< n$ each bracket $\equiv 0$

$$\therefore\; P(0) - nC_1P(y) + nC_2P(2y) - \cdots \equiv 0. \qquad \text{Theorem (3)}$$

If $p=n$ all the brackets vanish except the last, which becomes

$$S_o[0^n - nC_1 y^n + nC_2 2^n y^n - \cdots + (-1)^n n^n y^n] = (-y)^n n! \quad \text{Theorem (4)}$$

making $a=x$, $b=x-1$, $c=x-2$, etc., we have

$$(x)_n - nC_1(x+y)_n + nC_2(x+2y)_n - \cdots \equiv (-y)^n n!$$

$$(x)_p - nC_1(x+y)_p + \cdots\cdots \equiv 0. \quad \text{Theorems (7) and (8)}$$

Many other theorems can be obtained by varying the constants in (2) and (3).

§ 9. The Differential Equation of the n^{th} order

$$\left.\begin{aligned} &(q_o - p_o x)x^{n-1}\frac{d^n y}{dx^n} + (q_1 - p_1 x)x^{n-2}\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots \\ &\qquad \cdots\cdots + (q_{n-1} - p_{n-1}x)\frac{dy}{dx} - p_n y = 0 \end{aligned}\right\} \quad \text{(E)}$$

affords another very interesting analogy between powers, and products of related quantities.

When the constants p and q have the following values

$$p_n = a^n. \qquad\qquad q_{n-1} = \gamma^{n-1}$$

$$p_{n-1} = (a+1)^n - a^n. \qquad\qquad q_{n-2} = (\gamma+1)^{n-1} - \gamma^{n-1}$$

$$p_{n-2} = \frac{(a+2)^n}{2!} - \frac{(a+1)^n}{1!\,1!} + \frac{a^n}{2!}. \qquad\qquad \cdots\cdots\cdots\cdots$$

$$p_{n-3} = \frac{(a+3)^n}{3!} - \frac{(a+2)^n}{2!\,1!} + \frac{(a+1)^n}{1!\,2!} - \frac{a^n}{3!}. \qquad\qquad \cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots \qquad\qquad \cdots\cdots\cdots\cdots$$

$$p_o = 1 \qquad\qquad q_o = 1$$

The equation has a particular solution

$$y = A\left\{1 + \frac{a^n}{1!\,\gamma^{n-1}}x + \frac{a^n(a+1)^n}{2!\,\gamma^{n-1}(\gamma+1)^{n-1}}x^2 + \cdots\right\} \quad \cdots \quad \text{(F)}$$

When the constants in the differential equation have the following values

$$p_n \quad = (a)_n \cdot \qquad\qquad q_{n-1} = (\gamma)_{n-1}$$

$$p_{n-1} = (a+1)_n - (a)_n \cdot \qquad\qquad q_{n-2} = (\gamma+1)_{n-1} - (\gamma)_{n-1}$$

$$p_{n-2} = \frac{(a+2)_n}{2\,!} - \frac{(a+1)_n}{1\,!\,1\,!} + \frac{(a)_n}{2\,!}. \qquad\qquad \dots\dots\dots\dots$$

Etc. Etc.

The equation (E) has a solution

$$y = \mathrm{A}\left\{1 + \frac{(a)_n}{1\,!\,(\gamma)_{n-1}}x + \frac{(a)_n \cdot (a+1)_n}{2\,!\,(\gamma)_{n-1}(\gamma+1)_{n-1}}x^2 + \dots\right\} \qquad \dots \qquad \text{(G)}$$

The series (F) and (G) are particular cases of the Hypergeometric series of the n^{th} order.

On the Conditions that a given Straight Line may be a Normal to the Quadric Surface

$$(a, b, c, d, f, g, h, u, v, w)(x, y, z, 1)^2 = 0.$$

By R. H. Pinkerton, M.A.

Let the straight line be defined by the coordinates (α, β, γ) of a point on it and by its direction cosines l, m, n. It may be referred to as the line $(\alpha, \beta, \gamma, l, m, n)$. Write, for shortness, the equation to the quadric surface in the form $F(x, y, z) = 0$.

The line $(\alpha, \beta, \gamma, l, m, n)$ will be a normal to the quadric if it is perpendicular to either of the tangent planes to the quadric at the points where it cuts the quadric. The equation to this pair of tangent planes may be found as follows:

The line $(\alpha, \beta, \gamma, l, m, n)$ will cut the quadric in two points (x', y', z'), whose distances, r, from (α, β, γ) are the roots of the equation $\qquad F(\alpha + lr, \beta + mr, \gamma + mr) = 0,$

that is, of the equation

$$r^2(lL + mM + nN) + 2r(lP + mQ + nR) + F(\alpha, \beta, \gamma,) = 0 \quad \dots \quad (1)$$

in which

$$2L = \frac{dF(l, m, n)}{dl} - 2u, \quad 2M, \ 2N = \text{etc.},$$

and

$$2P = \frac{dF(\alpha, \beta, \gamma)}{d\alpha}, \qquad 2Q, \ 2R = \text{etc.}$$

The equation to the tangent plane to the quadric at the point (x', y', z') is

$$x(ax' + hy' + gz' + u) + y(hx' + ly' + fz' + v)$$
$$+ z(gx' + fy' + cz' + w) + ux' + vy' + wz' + d) = 0,$$

which, on writing $\alpha + lr, \beta + mr, \gamma + nr$ for x', y', z', becomes

$$r(xL + yM + zN + K) + (xP + yQ + zR + S) = 0 \quad \dots \quad (2)$$

in which $\qquad K = ul + vm + wn$

and $\qquad S = u\alpha + v\beta + w\gamma + d.$

If we now eliminate r between the equations (1) and (2), we shall obtain the required equation to the pair of tangent planes in the form

$$\begin{aligned}&(xP+yQ+zR+S)^2(lL+mM+nN)\\ &-2(xP+yQ+zR+S)(xL+yM+zN+K)(lP+mQ+nR)\\ &\qquad\qquad+F(\alpha,\ \beta,\ \gamma).(xL+yM+zN+K)^2=0.\end{aligned}$$

On multiplying this equation by $lL+mM+nN$, it will appear that it is equivalent to

$$\begin{aligned}&[(xP+yQ+zR+S)(lL+mM+nN)\\ &\qquad\qquad-(xL+yM+zN+K)(lP+mQ+nR)]^2\\ &=(xL+yM+zN+K)^2[(lP+mQ+nR)^2\\ &\qquad\qquad-F(\alpha,\ \beta,\ \gamma).(lL+mM+nN)],\end{aligned}$$

a form which clearly indicates two planes whose line of intersection is given by the equations

$$\begin{aligned}xP+yQ+zR+S&=0\\ xL+yM+zN+K&=0.\end{aligned}$$

In the equation just found write

U for $lL+mM+nN$,

V for $lP+mQ+nR$,

and ρ^2 for $(lP+mQ+nR)^2-F(\alpha,\ \beta,\ \gamma).(lL+mM+nN)$.

It follows from equation (1) that $\rho^2=0$ is the condition that the line $(\alpha,\ \beta,\ \gamma,\ l,\ m,\ n)$ may be a *tangent* line to the quadric, and that ρ is real or imaginary according as the line does or does not cut the quadric in real points.

On making the substitutions indicated we get as the equations to the tangent planes

$$\begin{aligned}&(xP+yQ+zR+S)U-(xL+yM+zN+K)V\\ &\qquad\qquad=\pm(xL+yM+zN+K)\rho,\end{aligned}$$

or
$$\begin{aligned}&x(PU-LV\pm L\rho)+y(QU-MV\pm M\rho)\\ &\qquad\qquad+z(RU-NV\pm N\rho)+(SU-KV\pm K\rho)=0,\end{aligned}$$

in which the signs in the ambiguities are to be taken all $+$ or all $-$.

Now the line $(\alpha, \beta, \gamma, l, m, n)$ is a normal to the quadric if it is perpendicular to either of these planes. Hence the line will be a normal if *one* of the two following sets of conditions is fulfilled:

$$(\mathrm{PU} - \mathrm{LV} + \mathrm{L}\rho)/l = (\mathrm{QU} - \mathrm{MV} + \mathrm{M}\rho)/m = (\mathrm{RU} - \mathrm{NV} + \mathrm{N}\rho)/n,$$
$$(\mathrm{PU} - \mathrm{LV} - \mathrm{L}\rho)/l = (\mathrm{QU} - \mathrm{MV} - \mathrm{M}\rho)/m = (\mathrm{RU} - \mathrm{NV} - \mathrm{N}\rho)/n.$$

It may be noticed that if *both* sets of conditions are fulfilled, the line is an axis of the quadric, for the line $(\alpha, \beta, \gamma, l, m, n)$ is then perpendicular to both of the tangent planes at the points where it meets the surface. Hence we arrive at the known conditions that the line $(\alpha, \beta, \gamma, l, m, n)$ may be an axis, viz.,

$$\mathrm{L}/l = \mathrm{M}/m = \mathrm{N}/n,$$

and

$$\mathrm{P}/l = \mathrm{Q}/m = \mathrm{R}/n.$$

Additional Note on Triangle Transformations.

By R. F. MUIRHEAD, M.A.

PART I.

The chief object of this Note is to develope some simple and rather interesting properties of the operators α, β, γ, referred to in my Note read at last meeting. I take for brevity the symbol μ to denote the compound operator

$$\alpha\beta\gamma \quad \text{or its equivalents} \quad \gamma\alpha\beta\,,\ \beta\alpha\gamma\,,\ \text{etc.}$$

1. If we apply μ to any function F, of $a, b, c, s, s_1, s_2, s_3, r, r_1, r_2, r_3, h_1, h_2, h_3, \Delta, R$; then, as I pointed out before, $\mu F = F$, if F be an *even* function of certain letters; but if F be an *odd* function, then $\mu F = -F$; and lastly, if $F \equiv F_1 + F_2$, where F_1 is odd and F_2 even, then $\mu F = -F_1 + F_2$.

2. The order of the operations α, β, γ successively applied, is immaterial.

If F be any function of the letters above mentioned, we have

$$\alpha^2 F = F, \text{ or } \alpha^2 = 1 \ \therefore\ \alpha = \frac{1}{\alpha}.$$

Again $\qquad \beta\gamma\gamma\beta = \beta\gamma^2\beta = \beta\beta = 1 \qquad \therefore\ \alpha\beta\gamma\gamma\beta = \alpha$

$$\therefore\ \mu\gamma\beta = \alpha.$$

Similarly $\qquad \mu\beta\gamma = \alpha \qquad \therefore\ \gamma\beta = \beta\gamma.$

Thus any two successive operators may be transposed without altering the result. Hence the order of any number of successive operations is immaterial.

3. Next, any succession of operations is reducible to one of eight operations. For by the last paragraph, any such succession is equivalent to $\alpha^m\beta^n\gamma^p$, where m, n, p are positive integers.

But $a^m = 1$ or a according as m is even or odd. Hence $a^m\beta^n\gamma^p$ reduces to one of the following eight operations:

$$1,\ a,\ \beta,\ \gamma,\ \beta\gamma,\ \gamma a,\ a\beta,\ a\beta\gamma;$$

which may be written

$$1,\ a,\ \beta,\ \gamma,\ \mu a,\ \mu\beta,\ \mu\gamma,\ \mu.$$

4. These form a "group" of operations, *i.e.*, any combination of them is equivalent to one or other of the eight.

And using the nomenclature explained by F. Klein in his "Lectures on the Icosahedron;" the *periodicity* of each operation (excepting the first) is 2; the most extended "*sub-groups*" are of the type $1,\ a,\ \mu a,\ \mu$; which again contains two *sub-groups* $1,\ a$ and $1,\ \mu$.

5 When the operation is performed on an *even* function, then $\mu = 1$, and the group reduces to $1,\ a,\ \beta,\ \gamma$; with $1,\ a$, etc., as sub-groups: and if on an *odd* function, then $\mu = -1$, and the group is $\pm 1,\ \pm a,\ \pm\beta,\ \pm\gamma$.

6. As pointed out at the end of the preceding Note, the angles ABC may occur in any functions which unaltered in absolute magnitude when $-A$ is changed into $-A + 2\pi$, etc., without modifying the above results; noting that A, B, C are reckoned along with r_1, r_1, etc., in counting the dimensions as *even* or *odd*.

PART II. (*Abstract*).

7. Reference was made to the following table for transforming a function of the general spherical triangle into the corresponding function of the *colunar* triangle opposite to A:

$$a,\ b,\ c,\ A,\ B,\ C,\ s,\ s-a,\ s-b,\ s-c,\ E,\ A-E,$$
$$B-E,\ C-E,\ L,\ N,\ n,\ R,\ R_a,\ R_b,\ R_c,\ r,\ r_a,\ r_b,\ r_c$$

become respectively

$$a,\ \pi-b,\ \pi-c,\ A,\ \pi-B,\ \pi-C,\ \pi-s+a,\ \pi-s,\ s-c,\ s-b,\ A-E,\ E,$$
$$C-E,\ B-E,\ L,\ N,\ n,\ R_a,\ R,\ R_c,\ R_b,\ r_a,\ r,\ r_c,\ r_b.$$

The notation used is that of Casey, who gives some of the transformations in his *Spherical Trigonometry*.

This, though similar to the transformation a in a plane triangle, is not strictly analogous.

8. Referring to the demonstration of the principle of the *transformation continue* as given by Mr Lemoine in the Report of the French Association for the Advancement of Science for 1891, Vol. II., p. 118, and ascribed by him to M. Laisant, it was pointed out that a somewhat simpler point of view is possible: from which it appears that many other kinds of transformations are equally valid. As interesting cases, the following were mentioned:

(1) ABC changed to $\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}, \frac{\pi}{2} - \frac{C}{2}$ and a into $a \operatorname{cosec} \frac{A}{2}$, etc., which corresponds to changing from the given triangle to that whose vertices are the excentres of ABC.

(2) s_1, s_2, s_3 changed to r_1, r_2, r_3.

Some properties of both were mentioned.

9. The suggestion made at the close of the preceding Note was verified by the example

$$\sin \frac{A}{2} = \sqrt{\frac{s_2 s_3}{bc}}$$

in which, after the transformation a, the square root must be taken *negatively*.

Added Note.—Dr Mackay having pointed out that α, β, γ have already been appropriated to denote certain quantities connected with the triangle, the author suggests the symbols t_1, t_2, t_3 to replace α, β and γ, in cases where these symbols have other meanings already ascribed to them.

Examples of a Method of Developing Logarithms and the Trigonometrical Functions without the Calculus by means of their Addition Formulae and Indeterminate Coefficients.

By JOHN JACK, M.A.

[ABSTRACT.]

The convergence of the series is *assumed*.

The method consists in assuming that the function is equal to a certain power series with undetermined coefficients, substituting these series in the addition formula. This gives an identity.

Ex. gr. $\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})$

$$\therefore\quad \left.\begin{array}{l} a_1x+a_2x^2+a_3x^3+\cdots \\ +a_1y+a_2y^2+a_3y^3+\cdots \end{array}\right\} = \left\{\begin{array}{l} a_1(x\sqrt{1-y^2}+y\sqrt{1-x^2})+a_2(x\sqrt{1-y^2}+y\sqrt{1-x^2})^2 \\ \qquad +a_3(a\sqrt{1-y^2}+y\sqrt{1-x^2})^3+\cdots \end{array}\right.$$

Picking out coefficient of y, we get

$$a_1=a_1+\text{function of } x.$$

Now this function of x must $=0$, and therefore the coefficients of the powers of x must each $=0$. From this it can be inferred that the function contains only odd powers of x, and the coefficients can easily be determined. The inverse function can be developed in the same way, and in the case of $\sin x$ or $\cos x$ with greater ease and completeness. I have found the development of $\text{sinam}x$, $\text{cosam}x$, and $\Delta\text{am}x$ and $\text{sinam}^{-1}x$ in the same way. The n^{th} term in the expansion of $\tan x$ is not given by this plan, that of $\sin^{-1}x$ can be inferred by *induction*. No. 2 has been

done of course in practically the same way, but is given on account of its intimate connection with No. 1. I give

(1) $\log x$	(3) $\sin^{-1}x$	(5) $\tan^{-1}x$
(2) $\log^{-1}x$ or e^x	(4) $\sin x$, $\cos x$	(6) $\tan x$.

The method seems symmetrical and quite elementary. The analogy between sin, cos, tan, and sinh, cosh, tanh, can be readily seen without at all using the imaginary i, by developing by this plan.

1. To develop $\log\overline{1+x}$ in a series of powers of x

$$\log(1+x)(1+y) = \log(1+x) + \log(1+y)$$

Let
$$\log(1+x) = \phi(x)$$

$$\therefore \quad \phi(x) + \phi(y) = \phi(x + y + xy)$$
$$= \phi(x + y\overline{1+x})$$

Let
$$\phi(x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$\therefore \quad \left.\begin{matrix} a_1x + a_2x^2 + a_3x^3 + \cdots \\ + a_1y + a_2y^2 + a_3y^3 + \cdots \end{matrix}\right\} \equiv \left\{\begin{matrix} a_1(x + y\overline{1+x}) + a_2(x + y\overline{1+x})^2 \\ + a_3(x + y\overline{1+x})^3 + \cdots \end{matrix}\right.$$

Pick out the coefficient of y.

$$\therefore \quad a_1 = (1+x)(a_1 + 2a_2x + 3a_3x^2 + \cdots)$$
$$= a_1 + (a_1 + 2a_2)x + (2a_2 + 3a_3)x^2 + \cdots$$

and the coefficients of x must vanish

$$\therefore \quad a_1 + 2a_2 = 0$$
$$2a_2 + 3a_3 = 0$$
$$3a_3 + 4a_4 = 0$$
$$4a_4 + 5a_5 = 0$$

and so on.

$$\therefore \quad a_1 = -2a_2 = 3a_3 = -4a_4 = 5a_5 = -, \ldots$$

$$\therefore \quad a^2 = -\frac{a_1}{2}, \; a_3 = \frac{a_1}{3}, \; a_4 = -\frac{a_1}{4}, \; a_5 = \frac{a_1}{5}, \text{ and so on}$$

$$\therefore \qquad \phi(x) = a_1\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\cdots\right)$$

$$\therefore \qquad \log(1+x) = a_1\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\cdots\right)$$

and a_1 must be determined otherwise.

The other expansions are given in abstract.

2. To find the number corresponding to a logarithm, or to develop $log^{-1}x$.

Taking as before $\qquad \phi(x) = \log(1+x)$

$$\phi(x) + \phi(y) = \phi(x + y + xy)$$

Let $\qquad \phi(x) = u \qquad \therefore \quad x = \phi^{-1}(u)$

$\qquad \phi(y) = v \qquad \therefore \quad y = \phi^{-1}(v)$

$$\therefore \qquad u + v = \phi\{\phi^{-1}(u) + \phi^{-1}(v) + \phi^{-1}(u)\phi^{-1}(v)\}$$

$$\therefore \qquad \phi^{-1}(u+v) = \phi^{-1}(u) + \phi^{-1}(v) + \phi^{-1}(u)\cdot\phi^{-1}(v)$$

Let $\qquad \phi^{-1}(\ ) = a_1(\ ) + a_2(\ )^2 + a_3(\ )^3 + \cdots$

Insert these expansions in the equation just given, and pick out the coefficients of v.

From the identity so obtained in powers of u, we get, by equating coefficients of like powers, the required relations between the constants $a_1 a_2 a_3 \cdots$, and finally

$$x = \phi^{-1}u = a_1 u + \frac{(a_1 u)^2}{\lfloor 2} + \frac{(a_1 u)^3}{\lfloor 3} + \frac{(a_1 u)^4}{\lfloor 4} + \frac{(a_1 u)^5}{\lfloor 5} + \cdots$$

$$\therefore \qquad 1 + x = 1 + a_1 u + \frac{(a_1 u)^2}{\lfloor 2} + \frac{(a_1 u)^3}{\lfloor 3} + \cdots$$

Now $\log\overline{1+x} = u$ and if a is the base

$$a^u = 1 + x$$

$$\therefore \qquad a^u = 1 + \frac{(a_1 u)}{\lfloor 1} + \frac{(a_1 u)^2}{\lfloor 2} + \frac{(a_1 u)}{\lfloor 3} + \cdots$$

and a_1 must be otherwise determined.

3. Required the development of $\sin^{-1}x$. By similar treatment of the identity

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

we get

$$\sin^{-1}x = a_1 \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{3}{2^3}\frac{x^5}{5} + \frac{5}{2^4}\frac{x^7}{7} + \frac{35}{2^7}\frac{x^9}{9} + \frac{63x^{11}}{2^8 11} + \frac{231x^{13}}{2^{10} 13} + \cdots\right)$$

$$= a_1\left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots\right)$$

and a_1 is otherwise found to be 1.

$\sin^{-1}x$ is thus found to be an odd function of x.

4. The development of $\sin u$, $\cos u$. This is got from the identity

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

or

$$u + v = \sin^{-1}(\sin u \cos v + \cos u \sin v.$$

It is shown, first, that $\sin u$ is an odd function of u, and $\cos u$ an even function of u. The series are then assumed, and the coefficients evaluated as above.

5. The development of $\tan^{-1}x$. This is got from the identity

$$\tan^{-1} + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

It is first established that $\tan^{-1}x$ is an odd function of x, and then the series is assumed and the coefficients evaluated in the usual way.

6. The development of $\tan u$. Here we have

$$\tan u + \tan v = \tan(u+v) - \tan u \tan v \tan(u+v).$$

Assume the series involving odd powers, and proceed as above.

The paper ended with an expansion in terms of arcs of small tangents, for calculating π.

Fifth Meeting, March 8th, 1895.

John M'Cowan, Esq., M.A., D.Sc., President, in the Chair.

Some Formulae in connection with the Parabolic Section of the Canonical Quadric.

By Charles Tweedie, M.A., B.Sc.

§ 1. I have ventured to bring the formulae of this paper before the Society, as I have been unable to find reference to them in any text-book or any original contribution to mathematical literature which I have come across. I confine my attention completely to the central surface, as the corresponding formulae for the paraboloids are very readily deduced by a similar process.

§ 2. Let the equation to the quadric be

$$\frac{x^2}{\alpha}+\frac{y^2}{\beta}+\frac{z}{\gamma}=1 \qquad \ldots \qquad \ldots \qquad (1)$$

where $\alpha\,\beta\,\gamma$ are not all of like sign.

Let the equation to a plane parabolic section be

$$lx+my+nz=p \qquad \ldots \qquad \ldots \qquad (2)$$

As the parallel central plane section must also be parabolic and touch the asymptotic cone $\Sigma\frac{x^2}{\alpha}=0$, we have the conditions

$$l^2+m^2+n^2=1 \qquad \ldots \qquad \ldots \qquad (\mathrm{I})$$

$$\alpha l^2+\beta m^2+\gamma n^2=0 \qquad \ldots \qquad \ldots \qquad (\mathrm{II})$$

Let $$\alpha^2l^2+\beta^2m^2+\gamma^2n^2=\mathrm{S} \qquad \ldots \qquad \ldots \qquad (\mathrm{III})$$

Then S can not vanish for real values of $l\,m\,n$, and the following relations may be established:

$$S=\Sigma a^2l^2=\Sigma m^2n^2(\beta-\gamma)^2=\sqrt{-\Sigma m^2n^2\beta\gamma(\beta-\gamma)^2} \quad \dots \quad \dots \quad (A)$$

$$=\beta m^2(\beta-a)+\gamma n^2(\gamma-a)=\text{etc.}=n^2(\gamma-a)(\gamma-\beta)-a\beta=\text{etc.}$$

$$\beta m^2(\beta-a)^2+\gamma n^2(\gamma-a)^2=S(\gamma+\beta-a)+a\beta\gamma,\ \text{etc.} \quad \dots \quad \dots \quad (B)$$

These might be added to, but they are all that are made use of in what follows.

Vertex.

§ 3. The coordinates of the vertex may be found by finding the equation of the plane in which the vertices of the parallel parabolic sections lie (just as for elliptic or hyperbolic sections), and solving for x, y, z with the aid of (1) and (2).

Now the axis of the parabola is parallel to what would be the diameter of the section as on the quadric, and therefore its direction cosines are $(\rho a l,\ \rho\beta m,\ \rho\gamma n)$, where ρ^2 is therefore equal to $1/S$.

If (x, y, z) be the vertex, the tangent to the parabola is perpendicular to $(\rho a l)$

i.e., $(\rho a l)$ is perpendicular to the direction given by the intersection of

$$lx+my+nz=p$$

and $$\frac{xx_1}{a}+\frac{yy_1}{\beta}+\frac{zz_1}{\gamma}=1$$

and therefore the vertex satisfies the equation

$$\Sigma\frac{x}{al}(\beta-\gamma)=0 \quad \dots \quad \dots \quad \dots \quad (3)$$

Solving (2) and (3) for y and z in terms of x, and remembering (A), we deduce

$$y/\beta m=x/al-p(a-\beta)/S \quad \dots \quad \dots \quad (4)$$

$$z/\gamma n=x/al-p(a-\gamma)/S \quad \dots \quad \dots \quad (5)$$

Substitute these values in (1); we have a quadratic in x, which, however, must have one infinite root, and which in virtue of (II) and (A) reduces to

$$2px/al = 1 - p^2\{\beta m^2(a-\beta)^2 + \gamma n^2(a-\gamma)\}^2/S^2$$

$$= 1 - \frac{p^2}{S}(\gamma + \beta - a) - a\beta\gamma p^2/S^2 \qquad \text{by (B).} \quad \ldots \quad (6)$$

Similarly

$$\frac{2py}{\beta m} = 1 - p^2(a + \gamma - \beta)/S - a\beta\gamma p^2/S^2. \quad \ldots \quad (7)$$

$$\frac{2pz}{\gamma n} = 1 - p^2(a + \beta - \gamma)/S - a\beta\gamma p^2/S^2. \quad \ldots \quad (8)$$

These are the coordinates of the vertex.

Parameter.

§ 4. Let V be the vertex, F the focus, FL the semi-latus rectum.

If (λ) be the direction of the tangent at the vertex, it will also be that of FL

Now (λ) is perpendicular to (l) of plane and to (al) of the axis

$$\left.\begin{aligned} \therefore\quad & \lambda l + \mu m + \nu n = 0 \\ \text{and}\quad & \lambda al + \mu\beta m + \nu\gamma n = 0 \end{aligned}\right\} \quad \ldots \quad (9)$$

and $\therefore\ \lambda : \mu : \nu = mn(\beta - \gamma) : nl(\gamma - a),\ lm(a - \beta)$

and $\therefore\ \lambda = \dfrac{mn(\beta-\gamma)}{\sqrt{\Sigma m^2n^2(\beta-\gamma)^2}} = \rho mn(\beta - \gamma)$, etc. ... (10)

Let 4π be the latus rectum then the coordinates of F are, if V be (xyz),

$$x + \pi\rho al,\quad y + \pi\rho\beta m,\quad z + \pi\rho\gamma n$$

and L is $(x + \pi\rho al + 2\pi\rho mn\overline{\beta - \gamma})$.

But L is on the quadric, and V is on the quadric

$$\therefore \quad \Sigma\frac{1}{a}\{x+\pi\rho al+2\pi\rho mn(\beta-\gamma)\}^2=1$$

$$\therefore \quad 2\pi\rho\left\{\Sigma lx+4\Sigma\frac{mnx}{a}(\beta \quad \gamma)\right\}$$
$$+\pi^2\rho^2\left\{\Sigma al^2+4lmn\Sigma(\beta-\gamma)+\Sigma\frac{m^2n^2(\beta-\gamma)^2}{a}\right\}=0$$

which by (2) and (3) reduces to

$$2\pi\rho p+\frac{4\pi^2\rho^2}{a\beta\gamma}\Sigma m^2n^2\beta\gamma(\beta-\gamma)^2=0$$

and $\therefore$
$$\begin{aligned}4\pi &= -2a\beta\gamma p/(-S^2)\rho\\ &= 2a\beta\gamma p\rho^3 \quad \text{by (A).} \qquad \ldots \qquad (11)\end{aligned}$$

So that for a system of parallel parabolic sections the parameter is not constant, as I had at first supposed, but varies directly as the distance of the section from the origin.

For the central section itself $p=0$, *i.e.* the parameter vanishes. The section then reduces to a pair of parallel lines for the hyperboloid of one sheet, which are coincident asymptotes in the two-sheeted surface. In the former there are, therefore, an infinite number of pairs of parallel generators, and, from the present analysis, one might infer that the parameter of a pair of parallel lines, considered as the limiting case of a parabola, is zero.

Focus.

§ 5. The coordinates of the focus are given by

$$\xi=x+\pi\rho al, \text{ etc.}$$

They are therefore given by

$$\left.\begin{aligned}\frac{2p\xi}{al}&=1-\frac{p^2}{S}(\gamma+\beta-a)\\ \frac{2p\eta}{\beta m}&=1-\frac{p^2}{S}(\gamma+a-\beta)\\ \frac{2p\zeta}{\gamma n}&=1-\frac{p^2}{S}(a+\beta-\gamma).\end{aligned}\right\} \qquad (12)$$

If we consider l, m, and n as connected by the equations (I) and (II) these contain implicitly equations (2) and (3).

For a system of parallel sections the locus of the foci is a plane curve, and the orthogonal projection on a reference plane as obtained by eliminating p between any two of the equations (12) is found to be a hyperbola whose centre is at the origin. Hence the locus of the foci of a series of parallel parabolic sections $\Sigma lx = p$ is a hyperbola whose centre is at the origin, and which lies in the plane

$$\Sigma \frac{x}{al}(\beta - \gamma) = 0.$$

Moreover $\Sigma lx = 0$ and $\Sigma \frac{x}{al}(\beta - \gamma) = 0$ are conjugate planes, and hence the foci for any system of parallel plane sections parallel to $\Sigma lx = 0$ is a conic lying in a plane conjugate to $\Sigma lx = 0$.

§ 6. The equation $\Sigma \frac{x}{al}(\beta - \gamma) = 0$ might be considered as given along with the equation $\Sigma al^2 = 0$; then we know that in it lies a hyperbola which is the locus of the foci of a system of parabolic sections. To find such a system, all we have to do is to take the diameter conjugate to the given plane, and draw the planes passing through it and touching the asymptotic cone. There are two such planes, real or coincident, since one is $\Sigma lx = 0$. They are not coincident since that would require that the diameter in question $\left(\frac{\beta - \gamma}{l}\right)$ should be a generator of the cone and $\therefore \Sigma m^2 n^2 \beta\gamma(\beta - \gamma)^2 - 0$, *i.e.*, $S = 0$ which is impossible.

There are then two systems having their foci lying in the same plane conjugate to both (these three planes are not a conjugate system).

§ 7. The plane $\Sigma \frac{x}{al}(\beta - \gamma) = 0$ being conditioned by $\Sigma al^2 = 0$ must therefore envelope a cone of the fourth class, whose equation is

$$\Sigma\{\beta\gamma(\beta - \gamma)^2 x^2\}^{\frac{1}{3}} = 0 \qquad \ldots \qquad \ldots \qquad (13)$$

and which is therefore of the sixth degree.

I omit the discussion of this surface, but the following may be noted:—

All the foci of the parabolic sections of a quadric will lie on a surface whose equation may be found by eliminating l, m, n, p from

any five independent equations involving them. This surface would seem to be of a fairly high degree, but, whatever its degree, every plane section whose plane is at the same time tangent to the cone (13) must be a degenerate curve, part or whole of whose real intersection consists of two concentric hyperbolas, with centre at the origin.

Surface of Revolution.

§ 8. For the surface of revolution, the formulae are much simpler. Suppose that $\beta = \alpha$, then $1/\rho^2 = S = -\alpha\gamma$, and $\therefore$ S is constant.

Equations (I) and (II) become

$$l^2 + m^2 + n^2 = 1 \quad \ldots \quad .. \quad \ldots \quad (I)'$$

$$\alpha(l^2 + m^2) + \gamma n^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (II)'$$

whence

$$\left.\begin{aligned} n^2 &= -\alpha/(\gamma - \alpha) \\ l^2 + m^2 &= \gamma/(\gamma - \alpha) \end{aligned}\right\} \quad \ldots \quad (14)$$

so that n^2 is constant, as is otherwise obvious.

The parameter

$$4\pi = 2\alpha\beta\gamma p\rho^3 \qquad \text{becomes}$$

$$4\pi = 2\alpha\beta\gamma p \frac{1}{\sqrt{-\alpha^3\gamma^3}} = \frac{2\alpha p}{\sqrt{-\alpha\gamma}} = 2p\sqrt{-\frac{\alpha}{\gamma}}, \quad \ldots \quad (15)$$

and therefore varies directly as the distance of the section from the origin.

The coordinates of the focus are given by

$$\frac{2p\xi}{\alpha l} = 1 - \frac{p^2}{-\alpha\gamma}\gamma = 1 + \frac{p^2}{\alpha} \quad \ldots \quad \ldots \quad (16)$$

$$\frac{2p\eta}{\alpha m} = \quad \ldots \quad = 1 + \frac{p^2}{\alpha} \quad \ldots \quad \ldots \quad (17)$$

$$\frac{2p\zeta}{\gamma n} = \quad 1 + \frac{p^2}{\alpha\gamma}(2\alpha - \gamma) \quad \ldots \quad \ldots \quad (18)$$

so that, (n being constant), when p is constant so also is ζ.

§ 9. Equations (16) and (17) give $l\eta - m\xi = 0$, and hence the cone of equation (13) shrinks up into the z-axis for the surface of revolution.

From the same equations we deduce

$$\frac{\xi}{l} = \frac{n}{m} = \sqrt{\frac{\xi^2 + \eta^2}{l^2 + m^2}} = cr, \text{ say, where}$$

$$r^2 = \xi^2 + \eta \quad \text{and} \quad c^2 = 1/(l^2 + m^2) = (\gamma - \alpha)/\gamma.$$

Hence, substituting in (16) and (18), we deduce

$$p^2 - 2pcr + \alpha = 0 \qquad \ldots \qquad \ldots \qquad (19)$$

$$p^2(2\alpha - \gamma) - 2p\frac{\alpha\zeta}{n} + \gamma\alpha = 0 \qquad \ldots \qquad \ldots \qquad (20)$$

On eliminating p from these two equations, we obtain the locus of the foci of all the parabolic sections.

The eliminant in question is

$$\left(\gamma\alpha - \alpha\cdot\overline{2\alpha - \gamma}\right)^2 = \left(-2cr\alpha\gamma + \frac{2\alpha^2\zeta}{n}\right)\left(-2\frac{\alpha\zeta}{n} + 2cr\cdot\overline{2\alpha - \gamma}\right) \qquad (21)$$

and this, on replacing c and n by their equivalents in terms of α and γ, reduces to

$$\alpha(\gamma - \alpha) - \alpha\zeta^2 - r^2\gamma + 2\alpha r^2 = 2\alpha^2\zeta r/(1 - \alpha\gamma). \qquad \ldots \qquad (22)$$

Take the square on both sides, and put $x^2 + y^2$ for r^2, z for ζ, when we obtain the locus of the foci of all parabolic sections,—a surface of revolution of the fourth degree, and such that all sections passing through the z-axis split up into degenerate curves consisting of two concentric hyperbolas.

Some Suggestions in Mathematical Terminology.

By R. F. Muirhead, M.A.

[Abstract.]

1. To designate the line which bisects at right angles the join of two points A, B, the term *axis of A, B* is proposed. Reasons :—(1) brevity; (2) avoidance of the suggestion that the line joining AB is necessary in constructing it (important in teaching Geometrical Drawing); (3) two points, like any other pair of circles, have a *radical axis* which is the line in question.

(2) To designate the tangent of the angle of inclination to the horizontal, or to the x-axis, or to any straight line or direction of reference: *gradient* is proposed. Reasons :—(1) Present Engineering usage; (2) as compared with the word *slope* it has a more definite suggestion of the way it is to be measured, *i.e.*, by trigonometrical tangent, not by angular magnitude; (3) already in use by several authorities.

(3) For unit of *moment of a force* on the British system of units: *pound-foot*, and *poundal-foot* for the gravitational and absolute unit respectively.

(4) For inverse trigonometrical and hyperbolic functions, *nis, soc, nat, toc, ces, cesoc, nish, soch, nath, toch, cesh, cesoch*, in place of sin^{-1}, cos^{-1}, etc., and of the continental *arc sin*, *arc sin hyp*, etc.

A Suggestion for the Improvement of Mathematical Tables.

By W. J. MACDONALD, M.A.

If in Tables such as Chambers' the differences were given for a second instead of for a minute, they might be arranged (and used) as follows:

30°	Lsin	Diff.
0′	9 6989700	1″ 36
1′	9·6991887	2″ 72
2′	9·6994073	3″ 108
3′	9·6996258	4″ 144
4′	9·6998441	5″ 180
5′	9·7000622	6″ 216
6′	9·7002802	7″ 252
7′	9·7004981	8″ 288
8′	9·7007158	9″ 324
9′	9·7009334	
10′	9·7011508	

To find Lsin30°4′29″

$$\begin{array}{lr} \text{Lsin}30°4' = & 9{\cdot}6998441 \\ \text{correction for } 20'' & 720 \\ \text{,, \quad ,, } 9'' & 324 \\ \hline \text{Lsin}30°4'29'' = & 9{\cdot}6999485 \end{array}$$

To find the angle whose

$$\begin{aligned} L\sin \quad &= 9{\cdot}7005632 \\ L\sin 30°7' &= 9{\cdot}7004981 \\ \hline & \qquad\quad 651 \\ 10'' = & \qquad\quad 360 \\ \hline & \qquad\quad 291 \\ 8'' = & \qquad\quad 288 \end{aligned}$$

$\therefore$ the angle is $30°7'18''$.

The following Resolution, moved by Mr DUTHIE, was unanimously adopted :—

"Considering that in the tables of lineal and square measure the fractional measures—viz., $5\frac{1}{2}$ yds. = 1 pl., and $30\frac{1}{4}$ sq. yds. = 1 sq. pl.—are of no practical value, and in their premature appearance in arithmetical study involve a grievous and unnecessary burden in teaching, this Society appeal to the Scottish Education Department to exercise its authority with a view to their abolition in schools, and to this end to allow no questions involving their use to be set in examinations under the control of the Department."

Sixth Meeting, April 10*th*, 1895.

JOHN M'COWAN, Esq., M.A., D.Sc., President, in the Chair.

On the Operation of Division.

By JOHN M'COWAN, M.A., D.Sc.

Sur les cubiques gauches équilatères.

By CH. BIOCHE.

J'appelle *cubiques gauches équilatères* les cubiques qui ont trois asymptotes rectangulaires deux à deux. Ces cubiques gauches possèdent des propriétés qui rappellent les propriétés classiques de l'hyperbole équilatère.

1. Si on prend pour origine des coordonnées un point d'une cubique équilatère E et pour axes des parallèles aux asymptotes, les coordonnées des points de cette cubique peuvent s'exprimer par

$$\text{(E)} \qquad X = \frac{A}{t-\alpha} \qquad Y = \frac{B}{t-\beta} \qquad Z = \frac{C}{t-\gamma}$$

A, B, C, α, β, γ étant des constantes, et t un paramètre variable. En effet il est visible que ces équations représentent une cubique équilatère passant par l'origine et ayant ses asymptotes parallèles aux axes. Comme une cubique est déterminée par six points, pour montrer qu'on a les équations le plus générales il suffit de montrer qu'on peut disposer des constantes de façon que la courbe E passe

par deux points arbitrairement choisis $(X_1Y_1Z_1)$ $(X_2Y_2Z_2)$. On a alors trois groupes de deux équations telles que

$$A + aX_1 = t_1X_1$$
$$A + aX_2 = t_2X_2 .$$

Les deux équations que je viens d'écrire donnent A et a, si X_1 est différent de X_2. D'ailleurs on peut remarquer que si on avait $X_1 = X_2$, il y aurait une droite rencontrant la courbe en trois points, cette courbe ne serait donc pas une cubique gauche.

2. Supposons que l'on considère quatre points 1, 2, 3, 4 de la courbe E, correspondant à des valeurs t_1, t_2, t_3, t_4 du paramètre variable. Les coefficients directeurs de la corde (1, 2) sont proportionals à

$$\frac{A}{(t_1 - \alpha)(t_2 - \alpha)}, \quad \frac{B}{(t_1 - \beta)(t_2 - \beta)}, \quad \frac{C}{(t_1 - \gamma)(t_2 - \gamma)}$$

Donc la condition pour que deux cordes (1, 2), (3, 4) soit rectangulaires s'exprime par l'equation

$$\text{(I)} \quad \frac{A^2}{(t_1 - \alpha)(t_2 - \alpha)(t_3 - \alpha)(t_4 - \alpha)} + \frac{B^2}{(t_1 - \beta)(t_2 - \beta)(t_3 - \beta)(t_4 - \beta)} + \frac{C^2}{(t_1 - \gamma)(t_2 - \gamma)(t_3 - \gamma)(t_4 - \gamma)} = 0$$

L'interprétation de cette équation donne diverses propriétés géométriques.

3. D'abord remarquons qu'elle est symétrique par rapport aux quatre indices. D'ici il resulte que les six arêtes du tétraèdre (1, 2, 3, 4) sont orthogonales.

Donc, *si deux cordes d'une cubique équilatère sont orthogonales, les extrémités de ces cordes sont les sommets d'un tétraèdre à arêtes opposées orthogonales ;* ou autrement dit *les droites qui joignent les extrémités de ces cordes sont deux à deux orthogonales.* Sous cette dernière forme l'enoncé s'applique, sans modification, au cas de l'hyperbole équilatère.

4. L'équation(I) est du 2ᵉ degré par rapport à t_4, par example; autrement dit, si on se donne trois points 1, 2, 3, il y a deux points M et M′ qui peuvent former avec les trois premiers un tétraèdre à arêtes opposées orthogonales. Or on sait que chaque sommet d'un tel tétraèdre se projette au point de rencontre des hauteurs de la face opposée. Donc M et M′ sont sur la perpendiculaire élevée sur le plan 1, 2, 3, au point de rencontre des hauteurs du triangle correspondant. Donc *la corde d'une cubique équilatère qui est perpendiculaire à un plan, rencontre ce plan au point de concours des hauteurs du triangle formé par ses trois points d'intersection avec la cubique.*

Inversement, *le lieu des points de concours des hauteurs des triangles déterminés par la cubique sur des plans parallèles est la corde perpendiculaire à ces plans.*

En particulier on voit qu'il y a deux plans, parmi ceux qui sont parallèles à un plan donné qui coupent une cubique équilatère en trois points formant un triangle rectangle. Ce sont ceux qui passent par les extrémités de la corde correspondante. Si cette corde devient tangente il n'y a plus qu'un plan, c'est le plan normal.

Le théorème général qui précède peut s'énoncer encore de la façon suivante, *si l'on projette orthogonalement une cubique équilatère sur un plan le point double de la projection est le point de concours des hauteurs du triangle formé par les points où le plan de projection coupe la cubique.*

5. Par chaque point de la cubique on peut mener une infinité de systèmes de trois cordes rectangulaires. Cela peut se déduire de ce qui précède, ou plus simplement de cette remarque que tout cône contenant la cubique admet évidemment trois arêtes rectangulaires parallèles aux trois asymptotes, et par suite admet une infinité de pareils systèmes.

On sait que si un angle droit est inscrit dans une hyperbole équilatère, la corde qui joint les points d'intersection des côtes de l'angle avec la courbe est parallèle à la normale au sommet de l'angle droit. On a un théorème analogue pour les cubiques équilatères. *Si un trièdre trirectangle ayant son sommet sur une cubique équilatère a ses trois arêtes s'appuyant sur la cubique, le plan qui passe par les trois points où ces arêtes rencontrent la cubique est parallèle au plan normal au sommet.*

Il suffit de démontrer le théorème pour le cas où le sommet de l'angle est à l'origine puisque l'origine est un point quelconque de la courbe.

Soient $(X_1Y_1Z_1)$, $(X_2Y_2Z_2)$, $(X_3Y_3Z_3)$ les points où les arêtes du trièdre rencontrent la cubique. Ces arêtes étant deux à deux rectangulaires, on a trois relations de la forme

$$X_1X_2 + Y_1Y_2 + Z_1Z_2 = 0$$

ou
$$\frac{A^2}{(t_1 - \alpha)(t_2 - \alpha)} + \frac{B^2}{(t_1 - \beta)(t_2 - \beta)} + \frac{C^2}{(t_1 - \gamma)(t_2 - \gamma)} = 0$$

Or si l'on remarque que

$$\frac{A}{t_1 - \alpha} - \frac{A}{t_2 - \alpha} = \frac{(t_2 - t_1)A}{(t_1 - \alpha)(t_2 - \alpha)}$$

on voit que l'équation précédente peut s'écrire

$$A \cdot \frac{A}{t_1 - \alpha} + B \cdot \frac{B}{t_1 - \beta} + C \cdot \frac{C}{t_1 - \gamma} = A \cdot \frac{A}{t_2 - \alpha} + B \cdot \frac{B}{t_2 + \beta} + C \cdot \frac{C}{t_2 - \gamma}$$

ou
$$AX_1 + BY_1 + CZ_1 = AX_2 + BY_2 + CZ_2 .$$

Ces expressions sont évidemment égales à

$$AX_3 + BY_3 + CZ_3 .$$

Les trois points considerés sont donc à la même distance du plan

$$AX + BY + CZ = 0$$

or il est facile de vérifier que ce plan est le plan normal.

Isoperimetric $2^m n$-gons applied to finding $\frac{1}{\pi}$ concisely by a new construction.

By R. E. Anderson, M.A.

FIGURE 27.

1. Let AB be the half-side of any n-gon, OB its in-radius (r), and OA its circum-radius (R). Draw OA_1 to bisect $\angle AOB$ and $AA_1C \perp$ to it meeting OB in C. Then $A_1B_1 \parallel$ to AB is the half-side of a $2n$-gon having the same perimeter as the n-gon, OB_1 its in-radius (r_1), and OA_1 its circum-radius (R_1).

Since B_1 bisects BC and $\triangle OB_1A_1$ is similar to OA_1C

$$\left.\begin{aligned} 2OB_1 &= OB + OC = OB + OA, \quad \therefore\ 2r_1 = r + R \\ \text{and} \qquad OA_1^2 &= OB_1 \,.\, OC = OB_1 \,.\, OA, \quad \therefore\ R_1^2 = r_1R \end{aligned}\right\} \text{ and } \therefore$$

$$2r_2 = r_1 + R_1, \quad 2r_3 = r_2 + R_2, \quad 2r_4 = \text{etc.}, \quad 2r_m = r_{m-1} + R_{m-1} \quad \ldots \quad (a)$$

$$R_2^2 = r_2R_1, \quad R_3^2 = r_3R_2, \quad R_4^2 = \text{etc.}, \quad R_m^2 = r_mR_{m-1}, \quad \ldots \quad (b)$$

where $R_2 = OA_2 = OC_2$, $R_3 = OA_3 = OC_3$, etc., $r_2 = OB_2$, $r_3 = OB_3$, etc., the new points being got by drawing $A_1A_2C_1$, $A_2A_3C_2$, etc., respectively $\perp$ to the successive bisectors OA_2, OA_3, etc., and A_2B_2, A_3B_3, etc., $\parallel$ AB or A_1B_1.

Thus, given r and R, we find $r_m (= OB_m)$ and $R_m (= OC_m)$ the radii of a polygon of $2^m n$ sides which has the same perimeter as the original n-gon. The diagram shows (1) that as m increases the two points B_m and C_m approach nearer and nearer to an intermediate point K; and (2) that the line $OK (= k)$ is the radius of a circle having also the perimeter in question.

Choosing the simplest case, $n = 2$, then if the common peri-

meter $=2$ units, OB$(=r)$ vanishes, $R=OA=AB=\frac{1}{2}$ and Fig. 27 is modified to Fig. 28. Also, circumference of the circle $=2=2\pi k$

$\therefore\ k=\dfrac{1}{\pi}$. Then, applying (a) and (b), we have

$$\left.\begin{array}{ll} r_3=\cdot 314,\ 208,\ 718,\ 257,\ 8(7) & R_3=\cdot 320,\ 364,\ 430,\ 968 \\ r_4=\cdot 317,\ 286,\ 574,\ 613,\ \ldots & R_4=\cdot 318,\ 821,\ 788,\ 7\ldots \\ r_5=\cdot 318,\ 054,\ 181,\ 6(5) & R_5=\cdot 318,\ 437,\ 75\quad \ldots \end{array}\right\}, \ldots (c)$$

Thus for the $2^5 . 2$-gon the radii agree to only 3 places. When m is large the following results will greatly reduce the labour of finding k.

2. OA_2 bisects $\angle A_1OC_1$, $\therefore$ A_1C_1 bisects $\angle CA_1B_1$ and $CC_1>2B_1B_2$. Thus $BB_1>4B_1B_2$, $B_1B_2>4B_2B_3$, etc.,

$$\text{and } BB_1+B_1B_2+\text{ etc.}, >4(B_1B_2+B_2B_3+\text{etc.}),$$

i.e. $$BK>4B_1K \text{ or } k-r>4(k-r_1)$$

$\therefore$ finally $k-r_{m-1}>4(k-r_m)$ and for a close value

$$r_m<k<\tfrac{1}{3}(4r_m-r_{m-1}), \ \ldots \quad \ldots \quad \ldots \quad (d)$$

Thus by (c) $k=\frac{1}{3}(4r_6-r_5)=\frac{1}{3}(2R_5+r_5)=\cdot 318,\ 309,\ 89.$

FIGURE 28.

To find a similar relation between the circum-radii I draw $A_2D \parallel A_1C$ and A_2E bisecting $\angle C_1A_2D$. The four adjacent acute angles at A_2 are equal $\therefore$ $C_2C_1<C_1E<ED$, CD or $C_1D>2C_1C_2$. Thus $CC_1>4C_1C_2$, $C_1C_2>4C_2C_3$, etc., and, as before, CK or $R-k>4C_1K$ or $4(R_1-k)$. Finally $R_{m-1}-k>4(R_m-k)$ and for a close value

$$R_m>k>\tfrac{1}{3}(4R_m-R_{m-1}), \ \ldots \quad \ldots \quad \ldots \quad (e)$$

Thus by (c) without using r_6

$$\tfrac{1}{3}(4R_5-R_4)=\cdot 318,\ 309,\ 74,\ \ \tfrac{1}{3}(4r_5-r_4)=\cdot 318,\ 310,\ 05$$

$\therefore$ $$k=\text{arith. mean}=\cdot 318,\ 309,\ 89 \text{ as above.}$$

When r_m and R_m agree to p places $p-1$ more can be found correctly by treating the new circum-radii as if they were in-radii.

Hence a third method of contraction when k has to be found to a large number of decimals. Calling the radii (after R_m) $a_1, a_2, a_3 \ldots a_x$ we shall have

$$\left.\begin{aligned} 2a_3 &= a_1 + a_2 \\ 2a_4 &= a_2 + a_3 \\ &\text{etc.} \\ 2a_x &= a_{x-2} + a_{x-1} \end{aligned}\right\} \quad \begin{aligned} &\therefore \text{ adding these } x-2 \text{ equations we get} \\ &\qquad a_1 + 2a_2 = 2a_x + a_{x-1} \\ &\qquad\qquad\quad = 3a_x \qquad \text{very nearly} \end{aligned}$$

$$\therefore\ k = \tfrac{1}{3}(r_m + 2R_m), \quad \ldots \quad \ldots \quad \ldots \ {}^*(f)$$

without finding the $x - 3$ intervening terms.

3. A fourth contraction may be derived from (d), thus:

$$\begin{aligned} 4(k - r_m) &< k - r_{m-1} \text{ gives } u = \tfrac{1}{3}(4r_m - r_{m-1}) \\ 4(k - r_{m-1}) &< k - r_{m-2} \quad ,, \quad v = \tfrac{1}{3}(4r_{m-1} - r_{m-2}) \end{aligned}$$

and $\qquad 16(k - u) < k - v$

$\therefore$ for a close value $k = \frac{1}{15}(16u - v)$, $\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (g)$

Thus, using (c)

$$\begin{aligned} u &= \tfrac{1}{3}(4r_5 - r_4) = \cdot 318,\ 310,\ 050,\ 7 \\ v &= \tfrac{1}{3}(4r_4 - r_3) = \cdot 318,\ 312,\ 526,\ 7 \end{aligned}$$

$\therefore$ by (g)

$$k = \cdot 318,\ 309,\ 886, \text{ taking 9 places;}$$

whence $\qquad \dfrac{1}{k} = \pi = 3 \cdot 141,\ 592,\ 65\ (54)$ by division.

It may be noted that if in Fig. 28 a quadrant BKMN be drawn its arc is exactly measured by the line AB, and its area is exactly measured by the rectangle $BN \,.\, BB_1$.

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* This result is given by Gergonne (*Annales*, Vol. VI.) with a more complicated proof, and recently by MM. Rouché and Comberousse with a different proof still more intricate (*Traité de Géom.*, 6th ed., 1891). For knowledge of the latter I am indebted to John S. Mackay, LL.D.

On the use of the Hyperbolic Sine and Cosine in connection with the Hyperbola.

By Lawrence Crawford, M.A., B.Sc.

The excentric angle notation in the ellipse is extremely useful, and in part we can replace it by the hyperbolic sine and cosine in connection with the hyperbola.

Take the hyperbola $x^2/a^2 - y^2/b^2 = 1$, then the coordinates of any point on it may be written $a\cosh\phi$, $b\sinh\phi$, for $\cosh^2\phi - \sinh^2\phi = 1$. The objection to its use in all cases is that the hyperbolic cosine of an angle is always positive, so that $(a\cosh\phi, b\sinh\phi)$ can only represent any point on the branch on the positive side of the axis of y, for any point on the other branch we must take its coordinates as $(-a\cosh\phi, b\sinh\phi)$.

Figure 28.

Take up the discussion of conjugate diameters in this notation.

Take a series of parallel chords, joining, first, points on the same branch, and let QQ′ be one of them, to find the locus of their middle points.

The line joining the points $(x_1y_1)(x_2y_2)$ on the hyperbola has the equation

$$\frac{x(x_1+x_2)}{a^2} - \frac{y(y_1+y_2)}{b^2} = 1 + \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2}$$

$\therefore$ if we take Q as the point $(a\cosh\alpha, b\sinh\alpha)$, Q′ as $(a\cosh\beta, b\sinh\beta)$ the equation of QQ′ is

$$\frac{x}{a}(\cosh\alpha + \cosh\beta) - \frac{y}{b}(\sinh\alpha + \sinh\beta) = 1 + \cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta$$

i.e. $$2\frac{x}{a}\cosh\frac{\alpha+\beta}{2}\cosh\frac{\alpha-\beta}{2} - 2\frac{y}{b}\sinh\frac{\alpha+\beta}{2}\cosh\frac{\alpha-\beta}{2} = 1 + \cosh(\alpha-\beta)$$
$$= 2\cosh^2\frac{\alpha-\beta}{2}$$

i.e. $$\frac{x}{a}\cosh\frac{\alpha+\beta}{2} - \frac{y}{b}\sinh\frac{\alpha+\beta}{2} = \cosh\frac{\alpha-\beta}{2}.$$

If then we are to have a series of parallel chords we must join points, parameters a and β, such that

$$\frac{b}{a}\coth\frac{a+\beta}{2} \text{ is constant.}$$

i.e. such that $a+\beta$ is constant.

Join then points, parameters $\lambda+\mu$ and $\lambda-\mu$, keeping λ constant and varying μ, then we shall get a series of parallel chords, the gradient of which is

$$\frac{b}{a}\coth\frac{\lambda+\mu+\lambda-\mu}{2} \quad i.e.\ \frac{b}{a}\coth\lambda.$$

Now the middle point of the line joining the points,

$$(a\cosh\overline{\lambda+\mu},\ b\sinh\overline{\lambda+\mu}) \text{ and } (a\cosh\overline{\lambda-\mu},\ b\sinh\overline{\lambda-\mu})$$

$$\text{is } \{\tfrac{1}{2}a(\cosh\overline{\lambda+\mu}+\cosh\overline{\lambda-\mu}),\ \tfrac{1}{2}b(\sinh\overline{\lambda+\mu}+\sinh\overline{\lambda-\mu})\}$$

i.e. $$(a\cosh\lambda\cosh\mu,\ b\sinh\lambda\cosh\mu)$$

$\therefore$ this point lies on the line $y=\dfrac{b}{a}\tanh\lambda\,.\,x$

$\therefore$ the locus of the middle points of this series of parallel chords, gradient $\dfrac{b}{a}\coth\lambda$, is the line $y=\dfrac{b}{a}\tanh\lambda\,.\,x$

Let us draw now a series of chords parallel to this latter line, $y=\dfrac{b}{a}\tanh\lambda\,.\,x$, and find locus of middle points of them.

Take a line joining two points on opposite branches,

$$(a\cosh\gamma,\ b\sinh\gamma) \text{ and } (-a\cosh\delta,\ b\sinh\delta),$$

its equation is

$$\frac{x}{a}(\cosh\gamma-\cosh\delta)-\frac{y}{b}(\sinh\gamma+\sinh\delta)=1-\cosh\gamma\cosh\delta-\sinh\gamma\sinh\delta,$$

using the form we already quoted,

$$i.e.\ 2\frac{x}{a}\sinh\frac{\gamma+\delta}{2}\sinh\frac{\gamma-\delta}{2}-2\frac{y}{b}\sinh\frac{\gamma+\delta}{2}\cosh\frac{\gamma-\delta}{2}=1-\cosh(\gamma+\delta)$$

$$=2\sinh^2\frac{\gamma+\delta}{2}$$

i.e. $$\frac{x}{a}\sinh\frac{\gamma-\delta}{2}-\frac{y}{b}\cosh\frac{\gamma-\delta}{2}=\sinh\frac{\gamma+\delta}{2}$$

The gradient of this line is $\frac{b}{a}\tanh\frac{\gamma-\delta}{2}$, and we wish it to be $\frac{b}{a}\tanh\lambda$, so join points, parameters γ and δ, so that $\gamma=\nu+\lambda$, $\delta=\nu-\lambda$ and keep λ constant but vary ν, then we get a series of parallel chords, gradient $\frac{b}{a}\tanh\lambda$.

The middle point of the line joining the points

$$(a\cosh\overline{\nu+\lambda},\ b\sinh\overline{\nu+\lambda})\quad(-a\cosh\overline{\nu-\lambda},\ b\sinh\overline{\nu-\lambda})$$

is
$$(a\sinh\nu\sinh\lambda,\ b\sinh\nu\cosh\lambda)$$

$\therefore$ this point lies on the line $y=\frac{b}{a}\coth\lambda\,.\,x$, which is therefore the locus of the middle points of this series of parallel chords, but this line is parallel to our original series of parallel chords,

$\therefore$ we have the lines $y=\frac{b}{a}\tanh\lambda\,.\,x$, $y=\frac{b}{a}\coth\lambda\,.\,x$ bisect each chords parallel to the other.

Thus the product of the gradients of two conjugate diameters is

$$\frac{b^2}{a^2}.$$

The line $y=\frac{b}{a}\tanh\lambda\,.\,x$ cuts the original hyperbola in P, which is the point $(a\cosh\lambda,\ b\sinh\lambda)$, while $y=\frac{b}{a}\coth\lambda\,.\,x$ cuts the conjugate hyperbola, $x^2/a^2-y^2/b^2=-1$ in the point $(a\sinh\lambda,\ b\cosh\lambda)$, say D.

These simple expressions for the coordinates of P and D give readily the theorems for example that $CP^2-CD^2=a^2-b^2$, that PD is bisected by the asymptote, and that tangents at P and D to the original and conjugate hyperbola respectively meet on the same asymptote.

Taking also again Q as the point $(a\cosh\overline{\lambda+\mu},\ b\sinh\overline{\lambda+\mu})$, and Q′ as $(a\cosh\overline{\lambda-\mu},\ b\sinh\overline{\lambda-\mu})$, and V as the middle point of QQ′, we easily prove

$$QV^2 : CV^2-CP^2 :: CD^2 : CP^2$$

which gives the equation of the hyperbola referred to the two conjugate diameters CP, CD as axes.

Seventh Meeting, May 10th, 1895.

WM. PEDDIE, Esq., M.A., D.Sc., in the Chair.

Proof of a Theorem in Conics.

By R. F. MUIRHEAD, M.A.

I.

In text books of Plane Coordinate Geometry, two methods are usually given for investigating the condition that the general equation of the second degree:

$$\phi \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

may represent a pair of real or imaginary straight lines.

The first is by identifying ϕ with the product of two linear factors, say $\lambda\lambda' \equiv (lx + my + nz)(l'x + m'y + n'z)$.
Equating coefficients, and eliminating l, m, n, l', m', n', we get

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, \text{ or, Discriminant} = 0$$

as the condition required.

The second method consists in solving $\phi = 0$ as a quadratic equation in x, and deducing the condition that the expression in y and z under the radical sign, should be a perfect square.

This as before, gives the condition : Discriminant $= 0$.

We may note by the way that of these two methods, the former, strictly speaking, proves only the *necessity*, and the latter, only the *sufficiency* of the condition; so that the propositions proved are converse, one of the other.

The object of this Note is to point out a short way of performing the elimination required in the former method, by forming the determinant which is the product of the two zero determinants

$$\begin{vmatrix} l, & l', & o \\ m, & m', & o \\ n, & n', & o \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} l', & l, & o \\ m', & m, & o \\ n', & n, & o \end{vmatrix}$$

The product is the symmetrical determinant

$$\begin{vmatrix} ll' + l'l, & lm' + l'm, & ln' + l'n \\ ml' + m'l, & mm' + m'm, & mn' + m'n \\ nl' + n'l, & nm' + n'm, & nn' + n'n \end{vmatrix}$$

which is of course identically equal to zero.

But if ϕ is identical with $\lambda\lambda'$ the determinant is obviously the same as

$$8 \times \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Thus the discriminant of ϕ is zero if ϕ represents a pair of straight lines.

Of course $\lambda\lambda' = 0$ is the standard form when we have a pair of *real* straight lines; and can only represent an *imaginary* pair when some of the coefficients are imaginary. The standard form for a pair of imaginary lines (or point-ellipse) would be $\lambda^2 + \lambda'^2 = 0$, where $\lambda \equiv lx + my + nz$, etc.

In this case the identification with ϕ gives

$$a = l^2 + l'^2, \; f = mn + m'n', \text{ etc., etc.}$$

And the elimination of l, m, n, l', m', n' can here be performed by squaring the zero determinant

$$\begin{vmatrix} l & l' & o \\ m & m' & o \\ n & n' & o \end{vmatrix}$$

and substituting a for $l^2 + l'^2$, f for $mn + m'n'$, etc., in the result.

II.

It occurred to me recently that this method of getting the condition *discriminant* $=0$ by multiplying two determinants, might be capable of application to discuss the discriminant in the general case. I have only had leisure to make a beginning in this direction, and none to look up the literature of the subject; but the following results seem interesting, and are new to me.

Suppose the general expression ϕ put into the form

$$p\lambda^2+p'\lambda'^2+p''\lambda''^2,$$

where $pp'p''$ are constants and $\lambda \equiv lx+my+nz$, etc.; thus we have

$$a=pl^2+p'l'^2+p''l''^2,\ f=pmn+p'm'n'+p''m''n'',\ \text{etc., etc.}$$

and the discriminant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ is obviously

the product $\begin{vmatrix} l & l' & l'' \\ m & m' & m'' \\ n & n' & n'' \end{vmatrix} \times \begin{vmatrix} pl, & p'l', & p''l'' \\ pm, & p'm', & p''m'' \\ pn, & p'n', & p''n'' \end{vmatrix}$

which may be written

$$pp'p'' \times \begin{vmatrix} l & l' & l'' \\ m & m' & m'' \\ n & n' & n'' \end{vmatrix}^2$$

and this is $=p\,.\,p'.\,p''.\,\mathrm{NN'N''}\times$ twice area of triangle formed by the lines $\lambda=0,\ \lambda'=0,\ \lambda''=0$; where N, N′, N″, are the minors of $n,\ n',\ n''$.

This of course vanishes when the lines are concurrent, in which case ϕ is expressible as the sum of two squared linear terms; and also when $pp'p''=0$, *i.e.* when one at least of the squared terms is awanting.

The lines $\lambda=0,\ \lambda'=0,\ \lambda''=0$ form a self-conjugate triangle for the conic; and such triangles are triply infinite in number for a given conic. We get the same result as to the possible number of ways of expressing ϕ in the form $p\lambda^2+p'\lambda'^2+p''\lambda''^2$ by noting that

there are 8 independent ratios between the coefficients of the latter expression, and only 5 in ϕ.

Again, it appears that the discriminant may vanish in virtue of p'' being zero, in which case the value of λ'' might be anything whatever; in fact, it seems that in such a case, while two sides of a self-conjugate triangle must pass through the centre of the conic, the position of the third is quite indeterminate, a result which is obvious also from the geometrical point of view.

Theorems in the Products of Related Quantities.

By F. H. Jackson, M.A.

§ 1. Let $(x)_n$ denote the function

$$\underset{\kappa=\infty}{\mathrm{L}} \frac{(x-n+1)(x-n+2)\cdots\cdots(x-n+\kappa)}{(x+1)(x+2)\quad\cdots\cdots\quad(x+\kappa)} \cdot \kappa^n$$

then

$$\frac{(x+r)_n}{\lfloor r \, \lfloor 0} - \frac{(x+r-1)_n}{\lfloor r-1 \, \lfloor 1} + \frac{(x+r-2)_n}{\lfloor r-2 \, \lfloor 2} - \cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots + (-1)^r \frac{(x)_n}{\lfloor 0 \, \lfloor r} = \frac{n \cdot n-1 \cdot n-2 \cdots n-r+1}{\lfloor r}(x)_{n-r} \quad \ldots (1)$$

In Gamma Functions the above may be written.

$$\frac{\Gamma(x)}{\Gamma(x-n)} - {}_rC_1 \frac{\Gamma(x-1)}{\Gamma(x-n-1)} + {}_rC_2 \frac{\Gamma(x-2)}{\Gamma(x-n-2)} - \cdots\cdots\cdots\cdots$$

$$\cdots\cdots + (-1)^r \frac{\Gamma(x-r)}{\Gamma(x-n-r)} = (n)_r \frac{\Gamma(x-r)}{\Gamma(x-n)} \quad \ldots (2)$$

By using the theorem (1) I shall obtain a purely algebraical proof of the well-known theorem

$$F_1(\alpha, \beta, \gamma) = \frac{\Pi(\gamma-1) \, . \, \Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\Pi(\gamma-\beta-1)}$$

where Π denotes Gauss's Π Function and $F_1(\alpha, \beta, \gamma)$ denotes the Hypergeometric Series in which the element $x_x = 1$.

It can be deduced from (1) that

$$1 - {}_rC_1 \frac{(x-n)_1}{(x)_1} + {}_rC_2 \frac{(x-n)_2}{(x)_2} - \cdots\cdots = \frac{(n)_r}{(x)_r} \text{ a pretty analogy with} \ldots (3)$$

$$1 - {}_rC_1 \frac{(x-n)^1}{x^1} + {}_rC_2 \frac{(x-n)^2}{x^2} - \cdots\cdots = \frac{(n)^r}{(x)^r}$$

r is not necessarily an integer in (3) and (1).*

---

* See § 7.

§ 2. **A fundamental property of the function $(x)_n$ is**

$$(x)_n \times (x-n)_m = (x)_{n+m}$$

whence we get $(x)_{n-r} \times (x-n+r)_s \times (x+r-s)_{r-s} = (x+r-s)_n$

Now the $(s+1)^{th}$ term on the left side of (1) $= (-1)^s \dfrac{(x+r-s)_n}{\underline{|r-s}\,\underline{|s}}$

which may be written

$$(-1)^s \frac{(x)_{n-r}(x-n+r)_s \cdot (x+r-s)_{r-s}}{\underline{|r-s}\;\underline{|s}}.$$

Since $(x+r-s)_{r-s}$ —when r and s are both integers—may be written in the form

$$(x+1)(x+2)(x+3)\cdots(x+r-s) = (-1)^{r-s}(-x-1)_{r-s}$$

$$\therefore \text{ the } (s+1)^{th} \text{ term} = (-1)^r \frac{(x)_{n-r}(x-n+r)\;(-x-1)_{r-s}}{\underline{|r-s}\;\underline{|s}}$$

The expression on the left side of (1) may be written

$$(-1)^r \frac{(x)_{n-r}}{\underline{|r}} \left\{ (-x-1)_r + \frac{\underline{|r}}{\underline{|r-1}\,\underline{|1}} (-x-1)_{r-1}(x-n+r)_1 \right.$$
$$\left. + \frac{\underline{|r}}{\underline{|r-2}\,\underline{|2}} (-x-1)_{r-2}(x-n+r)_2 + \cdots + (-1)^r (x-n+r)_r \right\} \quad (4)$$

By Vandermonde's theorem* the expression with the large bracket

$$= (-x-1+x-n+r)_r = (r-n-1)_r$$

Expression (4) becomes

$$(-1)^r \frac{(x)_{n-r}}{\underline{|r}} (-n-1)_r = \frac{n \cdot n-1 \cdot n-2 \cdots n-r+1}{\underline{|r}} (x)_{n-r}$$

which proves theorem (1).

---

* See § 7.

§ 3. Now
$$(x)_n = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$$

$$\because\ \Gamma(x+1) = \underset{\kappa=\infty}{\mathrm{L}} \frac{1\cdot 2\cdot 3\cdot 4\cdots\kappa}{(x+1)(x+2)\cdots(x+\kappa)}\kappa^x$$

Replacing $(\ )_n$ by Gamma Functions, the theorem (1), after multiplication throughout by $\underline{|r}$, becomes

$$\frac{\Gamma(x+r+1)}{\Gamma(x+r-n+1)} - {}_rC_1\frac{\Gamma(x+r)}{\Gamma(x+r-n)} + {}_rC_2\frac{\Gamma(x+r-1)}{\Gamma(x+r-n-1)} - \cdots\cdots$$
$$\cdots\cdots + (-1)^r\frac{\Gamma(x+1)}{\Gamma(x-n+1)} = (n)_r\frac{\Gamma(x+1)}{\Gamma(x-n+r+1)} \quad (5)$$

substitute y for $x+r+1$, then (5) becomes

$$\frac{\Gamma(y)}{\Gamma(y-n)} - {}_rC_1\frac{\Gamma(y-1)}{\Gamma(y-n-1)} + {}_rC_2\frac{\Gamma(y-2)}{\Gamma(y-n-2)} - \cdots\cdots$$
$$\cdots\cdots + (-1)^r\frac{\Gamma(y-r)}{\Gamma(y-n-r)} = (n)_r\frac{\Gamma(y-r)}{\Gamma(y-n)}$$

Remembering that $\Gamma(y) = (y-1)\Gamma(y-1)$ on division throughout by $\dfrac{\Gamma(y)}{\Gamma(y-n)}$ we have

$$1 - {}_rC_1\frac{(y-n-1)_1}{(y-1)_1} + {}_rC_2\frac{(y-n-1)_2}{(y-1)_2} - {}_rC_3\frac{(y-n-1)_3}{(y-1)_3} + \cdots = (n)_r\frac{1}{(y-1)_r}$$

this may be written

$$1 - r\cdot\frac{(x-n)_1}{(x)_1} + \frac{r\cdot r-1}{2\,!}\,\frac{(x-n)_2}{(x)_2} - \frac{r\cdot r-1\cdot r-2}{3\,!}\,\frac{(x-n)_3}{(x)_3} + \cdots = \frac{(n)_r}{(x)_r} \quad (6)$$

analogous to the Binomial Expansion

$$1 - r\cdot\frac{(x-n)}{(x)} + \frac{r\cdot r-1}{2\,!}\,\frac{(x-n)^2}{(x)^2} - \cdots = \frac{(n)^r}{(x)^r}$$

The Expansion (5) has been obtained on the supposition that r is a positive integer; but it will be shown later to hold for negative and fractional values of r.

§ 4. To consider the expansion in general of $f(x+y)$ in the form $P_0 + P_1(x)_1 + P_2(x)_2 + \cdots + P_r(x)_r + \cdots$ where $P_0 \cdot P_1 \cdot P_2 \cdots$ are functions of y only or constants. Assume that $f(x+y)$ is capable of being expanded in a convergent series of the above form then

$$f(x+y) = P_0 + P_1(x)_1 + P_2(x)_2 + \cdots + P_r(x)_r + \cdots$$

By giving x the values $0 \cdot 1 \cdot 2 \cdot 3 \cdots$ in succession we obtain the following equations to determine $P_0 \cdot P_1 \cdot P_2 \cdots$

$$f(y) \quad = P_0$$

$$f(y+1) = P_0 + P_1$$

$$f(y+2) = P_0 + 2P_1 + 2 \cdot 1 \; P_2$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$f(y+r) = P_0 + r \cdot P_1 + r \cdot r-1 \; P_2 + \ldots \underline{|r} \cdot P$$

$$\cdots\cdots\cdots\cdots\cdots$$

From which we obtain

$$P_0 = \frac{f(y)}{\underline{|0}\;\underline{|0}}$$

$$P_1 = \frac{f(y+1)}{\underline{|1}\;\underline{|0}} - \frac{f(y)}{\underline{|0}\;\underline{|1}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_r = \frac{f(y+r)}{\underline{|r}\;\underline{|0}} - \frac{f(y+r-1)}{\underline{|r-1}\;\underline{|1}} + \frac{f(y+r-2)}{\underline{|r-2}\;\underline{|2}} - \cdots + (-1)^r \frac{f(y)}{\underline{|0}\;\underline{|r}}$$

$$\cdots\cdots\;\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

which is that

$$f(x+y) = \sum_{r=0}^{r=\infty} \sum_{s=0}^{s=r} (-1)^s \frac{f(y+r-s)}{\underline{|r-s}\;\underline{|s}} (x)_r \qquad \ldots \qquad \ldots \qquad (7)$$

subject to the convergence of the series.

§ 5. The expansion of $(x+y)_n$, n being unrestricted.

The coefficient of x_r will be

$$P_r \equiv \frac{(y+r)_n}{\underline{|n}\,\underline{|0}} - \frac{(y+r-1)_n}{\underline{|n-1}\,\underline{|1}} + \frac{(y+r-2)_n}{\underline{|n-2}\,\underline{|2}} - \cdots \cdots + (-1)^r \frac{(y)_n}{\underline{|0}\,\underline{|n}} \equiv \frac{n\cdot n-1\cdots n-r+1}{\underline{|r}} y_{n-r}$$

(by Theorem (1)).

$$\therefore \quad (x+y)_n = y_n + ny_{n-1}x_1 + \frac{n\cdot n-1}{\underline{|2}} y_{n-2}x_2 + \cdots \cdots + \frac{n\cdot n-1\cdots x-r+1}{\underline{|r}} y_{n-r}x_r + \cdots \cdots \qquad (8)$$

This is the generalised form of Vandermonde's Theorem; the proof depends, as will be seen on reference to § 2, No. 4, on Vandermonde's Theorem for positive integral values of the suffix.

To expand a^x in a series of form (7)

we have $$P_r = \frac{a^r}{\underline{|r}\,\underline{|0}} - \frac{a^{r-1}}{\underline{|r-1}\,\underline{|1}} + \cdots + (-1)^r \frac{a^0}{\underline{|0}\,\underline{|r}} \equiv \frac{(a-1)^r}{\underline{|r}}$$

$$\therefore \quad a^x = 1 + (a-1)(x)_1 + \frac{(a-1)^2}{\underline{|2}}(x)_2 + \cdots + \frac{(a-1)^r}{\underline{|r}}(x)_r + \cdots \cdots$$

this is a well known particular case of the Binomial Expansion.

To expand $\dfrac{1}{x+a}$.

we have $$P_r = \frac{1}{\underline{|r}}\left\{\frac{1}{a+r} - {}_rC_1\frac{1}{a+r-1} + {}_rC_2\frac{1}{a+r-2} - \cdots + (-1)^r\frac{1}{a}\right\}$$

$$= \frac{1}{\underline{|r}} \cdot \frac{\underline{|r}}{(a+r)(a+r-1)\cdots(a+1)a}\cdot$$

$$\therefore \quad \frac{1}{x+a} = \frac{1}{a} + \frac{(x)_1}{a\cdot a+1} + \frac{(x)_2}{a\cdot a+1\cdot a+2} + \cdots + \frac{(x)_r}{a\cdot a+1\cdots a+r} + \cdots$$

This is a special case of Vandermonde's Theorem for negative integral values of the suffix. The functions which can be expanded in series of form (7) seem very restricted in number.

§ 6. Writing

$$(x+y)_n = (y)_n + n \cdot (y)_{n-1}(x)_1 + \frac{n \cdot n-1}{\underline{2}\;\underline{|r}}(y)_{n-2}(x)_2 + \cdots$$

$$\cdots\cdots + \frac{n \cdot n-1 \cdots n-r+1}{\underline{|r}}(y)_{n-r}(x)_r + \cdots\cdots$$

divide both sides by $(y)_n$.

Then $$\frac{(x+y)_n}{(y)_n} = 1 + n \cdot \frac{(y)_{n-1}(x)_1}{(y)_n} + \frac{n \cdot n-1}{\underline{|2}\;\underline{|r}} \frac{(y)_{n-2}(x)_2}{(y)_n} + \cdots$$

Now it is easily seen that $$\frac{(y)_{n-1}}{(y)_n} = \frac{1}{y-n+1}$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{(y)_{n-r}}{(y)_n} = \frac{1}{(y-n+1)_r}$$

and $\dfrac{(x+y)_n}{(y)_n} = \dfrac{\Pi(x+y)}{\Pi(x+y-n)} \cdot \dfrac{\Pi(y-n)}{\Pi(x)}$ where Π denotes Gauss's Π Function. Therefore

$$\frac{\Pi(x+y) \cdot \Pi(y-n)}{\Pi(x+y-n)\,\Pi(x)} = 1 + n \cdot \frac{(x)_1}{(y-n+1)_1}$$

$$+ \frac{n \cdot n-1}{2\,!} \frac{(x)_2}{(y-n+1)(y-n+2)} + \cdots + \frac{n \cdots n-r+1}{r\,!} \frac{(x)_r}{(y-n+1)_r} + \cdots$$

Replacing n by $-\alpha$, x by $-\beta$, and $y-n+1$ by γ we have

$$\frac{\Pi(\gamma-\alpha-\beta-1) \cdot \Pi(\gamma-1)}{\Pi(\gamma-\beta-1)\Pi(\gamma-\alpha-1)} = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} + \frac{\alpha \cdot \alpha+1 \cdot \beta \cdot \beta+1}{1 \cdot 2 \cdot \gamma \cdot \gamma+1} + \cdots \tag{9}$$

$$= F_1(\alpha, \beta, \gamma)$$

§ 7. If in § 2, result (4), we had assumed the truth of Vandermonde's Theorem for unrestricted values of the suffix, Theorems (1), (2), and (3) would have been proved for all values of r. Since we have proved Vandermonde's Theorem for unrestricted values of the suffix, the proofs of §§ 2 and 3 may be repeated with r unrestricted. The use of $(-1)^r$ in § 2 can easily be avoided. When r is unrestricted, $(r)_r$ must be used instead of $\underline{|r}$.

Isogonals of a Triangle.

By J. S. MACKAY, M.A., LL.D.

DEFINITION.—*If two angles have the same vertex and the same bisector, the sides of either angle are isogonal* to each other with respect to the other angle.*

Thus the isogonal of AP with respect to ∠ BAC is the image of AP in the bisector of ∠ BAC. It is indifferent whether the bisector of the interior ∠ BAC be taken, or the bisector of the angle adjacent to it; the isogonal of AP remains the same.

It follows from the definition that

(1) The internal and the external bisectors of ∠ BAC are their own isogonals.

(2) The line joining the orthocentre of a triangle to any vertex is isogonal to the line joining the circumcentre to that vertex.

(3) Any internal median of a triangle is isogonal to the corresponding symmedian.

(4) The tangents to the circumcircle of ABC at A, B, C are isogonal to the external medians.

[The external medians are the parallels to the sides of a triangle drawn through the opposite vertices.

The reason for the giving of this name will be found in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I., p. 16 (1894).]

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* This terminology was proposed by Mr G. de Longchamps in his *Journal de Mathématiques Élémentaires*, 2nd series, Vol. V., p. 245 (1886).

§ 1.

(*a*) *If P, Q be any two points taken on a pair of lines isogonal with respect to angle BAC, the distances of P from AB, AC are inversely proportional* to those of Q from AB, AC.*

FIGURE 29.

If the quadrilateral AQ_2QQ_1 be revolved through two right angles round the bisector of $\angle B$ as an axis, it will become homothetic to the quadrilateral AP_1PP_2; therefore

$$PP_1 : PP_2 = QQ_2 : QQ_1$$

(*a'*) *If P, Q be any two points and if the distances of P from AB, AC be inversely proportional to those of Q from AB, AC, then AP, AQ are isogonal with respect to $\angle BAC$.*

This may be proved indirectly.

(1) *The points P_1 Q_1 Q_2 P_2 are concyclic*†

Since P_1P_2 Q_1Q_2 are antiparallel with respect to $\angle BAC$; therefore P_1 P_2 Q_1 Q_2 are concyclic.

(2) *The centre of the circle $P_1Q_1Q_2P_2$ is the mid point of PQ.*

For the perpendicular to P_1Q_1 at its mid point goes through the centre of the circle; and this perpendicular bisects PQ.

So does the perpendicular to P_2Q_2 at its mid point.

(3) *P_1P_2 is perpendicular to AQ*
and Q_1Q_2 ,, ,, ,, AP.

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* Sir James Ivory in Leybourn's *Mathematical Repository*, new series, Vol. I., Part II., p. 19 (1806). The mode of proof is due to Professor Neuberg. See his excellent memoir on the Recent Geometry of the Triangle in Rouché and Comberousse's *Traité de Géométrie*, First Part, p. 438 (1891).

† This and the two following theorems are due to Steiner. See Gergonne's *Annales*, XIX., 37-64 (1828), or Steiner's *Gesammelte Werke*, I., 191-210 (1881). The proof given of (1) is Professor Neuberg's. See the reference in the preceding note.

For AP is a diameter of the circumcircle of AP_1P_2; therefore the isogonal of AP with respect to $\angle P_1AP_2$ is the perpendicular * from A to P_1P_2

(4) The circumcentre of either of the triangles AP_1P_2 AQ_1Q_2 and the orthocentre of the other are collinear with the point A.

(5) *Triangle PP_1P_2 is inversely similar* † *to QQ_2Q_1.*

This follows from the demonstration of § 1; or it may be thus proved:

$$\angle PP_1P_2 = \angle PAP_2 = \angle QAQ_1 = \angle QQ_2Q_1.$$

Similarly $\angle PP_2P_1 = \angle QQ_1Q_2.$

(6) *If PP_1 QQ_2 meet at D*
and PP_2 QQ_1 „ „ E,
then AD, AE are isogonals with respect to $\angle BAC$.

FIGURE 30.

Join P_1Q_2 P_2Q_1.

Since P_1Q_1 P_2Q_2 are concyclic,
therefore $\angle AQ_2P_1 = \angle AQ_1P_2$
therefore their complements are equal
that is $\angle P_1Q_2D = \angle P_2Q_1E.$
Similarly $\angle Q_2P_1D = \angle Q_1P_2E$;
therefore triangles P_1Q_2D, P_2Q_1E are similar;
therefore $P_1Q_2 : P_1D = P_2Q_1 : P_2E.$
Now triangles AP_1Q_2, AP_2Q_1 are similar;
therefore $AP_1 : P_1Q_2 = AP_2 : P_2Q_1.$
Hence $AP_1 : P_1D = AP_2 : P_2E$
and $\angle P_1AD = \angle P_2AE.$

The same result might be arrived at by revolving the quadrilateral AQ_2QQ_1 through two right angles round the bisector of $\angle BAC$.

* This mode of proof is given by Professor Fuhrmann in his *Synthetische Beweise planimetrischer Sätze*, p. 93 (1890).
† See Ivory's paper already cited, p. 20.

§ 2.

(*a*) *If ABC be a triangle, and if AP, AQ be isogonal with respect to A, then* *

$$BP \cdot BQ : CP \cdot CQ = AB^2 : AC^2$$

FIGURE 31.

About APQ circumscribe a circle, cutting AB, AC in F, E; join FE.

Because	$\angle BAP = \angle CAQ$
therefore	arc FP = arc EQ
therefore	FE is parallel to BC
therefore	$AB : BF = AC : CE$
therefore	$AB^2 : AB \cdot BF = AC^2 : AC \cdot CE$
therefore	$AB^2 : BP \cdot BQ = AC^2 : CP \cdot CQ$.

A second demonstration will be found in C. Adams's *Die merkwürdigsten Eigenschaften des geradlinigen Dreiecks*, p. 1 (1846), and a third in Professor Fuhrmann's *Synthetische Beweise*, p. 94 (1890).

(*a'*) *If ABC be a triangle and BC be divided at P and Q so that*

$$BP \cdot BQ : CP \cdot CQ = AB^2 : AC^2$$

then † *AP, AQ are isogonals with respect to A.*

This may be proved indirectly.

(1) If AQ be the internal or the external median from A, then $BQ = CQ$, and the theorem becomes ‡

$$BP : CP = AB^2 : AC^2.$$

* Pappus's *Mathematical Collection*, VI. 12. The same theorem differently stated is more than once proved in Book VII. among the lemmas which Pappus gives for Apollonius's treatise on *Determinate Section*. The proof in the text is taken from Pappus.

† In Pappus's *Mathematical Collection*, VI. 13, there is proved the theorem:

$$\text{If } BP \cdot BQ : CP \cdot CQ > AB^2 : AC^2$$

then $$\angle BAP > \angle CAQ.$$

‡ Adams (see the reference to him on this page) gives (1)-(4), (6), (8). His proof of (4) is different from that in the text.

(2) *If AQ be the internal or the external median from A and* $\angle$ *BAC be right, then AP is perpendicular to BC.*

FIGURES 32, 33.

$$\text{Since} \qquad \angle ACB = \angle CAQ = \angle BAP$$
$$\text{therefore } \angle ACB + \angle CAP = \angle BAP + \angle CAP$$
$$= \text{a right angle.}$$

(3) If AP and AQ coincide, then AP is either the internal or the external bisector of $\angle$ A, and the theorem becomes

$$BP^2 : CP^2 = AB^2 : AC^2$$

or

$$BP : CP = AB : AC$$

a known result, namely, Euclid VI. 3, or the cognate theorem.

(4) $$BP \cdot CP : BQ \cdot CQ = AP^2 : AQ^2.$$

This follows from the theorem of § 2 by considering APQ as the triangle and AB, AC as the isogonals.

(5) *If AP, AQ which are isogonal with respect to* $\angle$ *BAC meet the circumcircle of ABC in R, S, then* $AP \cdot AS = AQ \cdot AR$.

FIGURE 34.

For triangles ACR, AQB are similar

therefore $\quad AQ \cdot AR = AB \cdot AC.$

Similarly $\quad AP \cdot AS = AB \cdot AC.$

(6) RS is parallel to BC.

(7) The distances from the mid point of any side of a triangle to the points where two isogonals from the opposite vertex meet the circumcircle are equal.*

For the perpendicular which bisects BC bisects RS.

---

* Mr Emile Vigarié in the *Journal de Mathématiques Élémentaires*, 2nd series, IV. 59 (1885).

(8) If APR becomes the diameter of the circumcircle ABC then AQ becomes perpendicular to BC, and

$$AQ \cdot AR = AB \cdot AC,$$

a theorem of Brahmegupta's.

See Chasles's *Aperçu*, 2nd ed., pp. 420-447.

(9) If AP, AQ coincide, then AP becomes either the internal or the external bisector of $\angle$ A.
Hence in the first case

$$\begin{aligned} AB \cdot AC &= AP \cdot AS \\ &= AP \cdot PS + AP^2 \\ &= BP \cdot PC + AP^2 ; \end{aligned}$$

and in the second case

$$\begin{aligned} AB \cdot AC &= AP \cdot AS \\ &= AP \cdot PS - AP^2 \\ &= BP \cdot PC - AP^2. \end{aligned}$$

(10) *In triangle ABC, AP, AQ are isogonals with respect to A; through B draw BE parallel to AP meeting CA in E;*
" C " CF " " AQ " BA " F;
*then EF is antiparallel * to BC with respect to A.*

FIGURE 35.

For $\angle ABE = \angle BAP = \angle CAQ = \angle ACF$;
therefore the points E, B, C, F are concyclic.

The same thing would happen if BE, CF were drawn parallel to AQ, AP.

(11) *In triangle ABC, AP, AQ are isogonal; from P and Q perpendiculars are drawn to BC; these perpendiculars are intersected at D, E by a perpendicular to AB at B, and at D′, E′ by a perpendicular to AC at C. To prove* †

$$BD \cdot BE : CD' \cdot CE' = AB^4 : AC^4.$$

* Mr Emile Vigarié.

† Mr Emile Vigarié in the *Journal de Mathématiques Élémentaires*, 2nd series, IV. 224 (1885) says that this theorem was communicated to him by his friend Mr Th. Valiech.

FIGURE 36.

Draw AX perpendicular to BC.
The similar triangles BDP, BEQ, ABX give

$$BD : BP = AB : AX$$
$$BE : BQ = AB : AX$$

therefore
$$\frac{BD \cdot BE}{BP \cdot BQ} = \frac{AB^2}{AX^2}$$

Similarly
$$\frac{CD' \cdot CE'}{CP \cdot CQ} = \frac{AC^2}{AX^2}$$

therefore
$$\frac{BD \cdot BE}{CD' \cdot CE'} \cdot \frac{CP \cdot CQ}{BP \cdot BQ} = \frac{AB^2}{AC^2}$$

therefore
$$\frac{BD \cdot BE}{CD' \cdot CE'} \cdot \frac{AC^2}{AB^2} = \frac{AB^2}{AC^2}$$

(12) *If in* (11) *AQ be the median* from A,*
then $$BD : CD' = AB^3 : AC^3.$$

FIGURE 36.

For $BE : BQ = AB : AX$
and $CE' : CQ = AC : AX$;
therefore $BE : CE' = AB : AC$,
whence the result follows.

§ 3.

If three straight lines drawn through the vertices of a triangle are concurrent, their isogonals with respect to the angles of the triangle are also concurrent.‡

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* Mr Emile Vigarié in the *Journal de Mathématiques Élémentaires*, 2nd series, IV. 225 (1885).

† Steiner in Gergonne's *Annales*, xix. 37-64 (1828), or Steiner's *Gesammelte Werke*, I. 193 (1881). Ivory in his paper previously cited proves the theorem :

If the isogonals BO, BO′ meet the bisector of $\angle$ A at O, O′,
then $$BO : CO = BO' : CO';$$
and he adds as a corollary that CO, CO′ are isogonals with respect to C.

FIGURE 37.

Let BO, BO′ be isogonals with respect to B

and CO, CO′ ,, ,, ,, ,, C;

then AO, AO′ are ,, ,, ,, A.

Denote the distances of O from the sides by p_1 p_2 p_3 and those of O′ by q_1 q_2 q_3

Then $$p_1 q_1 = p_2 q_2 \text{ and } p_1 q_1 = p_3 q_3$$

therefore $$p_2 q_2 = p_3 q_3$$

therefore AO, AO′ are isogonals with respect to A.

Another demonstration will be found in C. Adams's *Eigenschaften des...Dreiecks*, pp. 7-8 (1846).

Points such as O, O′ determined by the intersection of pairs of isogonal lines will be called *isogonal points*, or simply *isogonals*, with respect to the triangle ABC.

They are sometimes* called *isogonally conjugate points*, or *isogonal conjugates*, but more frequently on the continent of Europe *inverse points* with respect to the triangle ABC.

The designation, *inverse points*, was suggested about the same time in Scotland and in France. See a paper read before the Royal Society of Edinburgh on 20th March 1865, by the Rev. Hugh Martin, and printed in their *Transactions*, xxiv. 37-52: and an article by Mr J. J. A. Mathieu in the *Nouvelles Annales*, 2nd series, IV. 393–407, 481-493, 529-537 (1865).

Perhaps the adoption of the nomenclature proposed by Mr G. de Longchamps in the *Journal de Mathématiques Élémentaires*, 2nd series, V. 109 (1886) would be advantageous.

(1) $\angle BOC + \angle BO'C = 180° + A.$

FIGURE 37.

$$\begin{aligned}
\text{For} \quad \angle \text{BOC} &= \text{A} + \text{ABO} + \text{ACO}, \\
&= \text{A} + \text{CBO}' + \text{BCO}', \\
\text{and} \quad \angle \text{BO}'\text{C} &= \text{A} + \text{ABO}' + \text{ACO}'; \\
\text{therefore} \quad \angle \text{BOC} + \angle \text{BO}'\text{C} &= 2\text{A} + \text{B} + \text{C}, \\
&= 180° + \text{A}.
\end{aligned}$$

* Professor J. Neuberg's *Mémoire sur le Tétraèdre*, p. 10 (1884).

(2) In triangle ABC, AP_1 BP_2 CP_3 are concurrent at O, and their isogonals AQ_1 BQ_2 CQ_3 are concurrent at O′.

FIGURE 36.

Suppose BP_2 BQ_2 to form one straight line
and CP_3 CQ_3 " " " " ";
then the points O O′ coincide.*

There are four cases.

(*a*) If BP_2 CP_3 bisect the interior angles B, C, then AP_1 bisects the interior angle A.

(*b*) If BP_2 CP_3 bisect the exterior angles B, C, then AP_1 bisects the interior angle A.

(*c*) If BP_2 bisects the interior angle B
and CP_3 " " exterior " C,
then AP_1 " " exterior " A.

(*d*) If BP_2 bisects the exterior angle B
and CP_3 " " interior " C,
then AP_1 " " exterior " A.

Hence the six bisectors of the angles of a triangle meet three by three in four points.

FIGURE 36.

(3) By considering AP_1Q_1 as the triangle, and AB, AC as the isogonals

$$BP_1 \cdot CP_1 : BQ_1 \cdot CQ_1 = AP_1^2 : AQ_1^2.$$

Similarly $$CP_2 \cdot AP_2 : CQ_2 \cdot AQ_2 = BP_2^2 : BQ_2^2,$$

and $$AP_3 \cdot BP_3 : AQ_3 \cdot BQ_3 = CP_3^2 : CQ_3^2;$$

therefore $$\frac{BP_1 \cdot CP_2 \cdot AP_3 \cdot CP_1 \cdot AP_2 . BP_3}{BQ_1 \cdot CQ_2 \cdot AQ_3 \cdot CQ_1 \cdot AQ_2 \cdot BQ_3} = \frac{AP_1^2 \cdot BP_2^2 \cdot CP_3^2}{AQ_1^2 \cdot BQ_2^2 \cdot CQ_3^2}.$$

Now $$BP_1 \cdot CP_2 \cdot AP_3 = CP_1 \cdot AP_2 \cdot BP_3$$

and $$BQ_1 \cdot CQ_2 \cdot AQ_3 = CQ_1 \cdot AQ_2 \cdot BQ_3;$$

therefore $$\frac{AP_1 \cdot BP_2 \cdot CP_3}{AQ_1 \cdot BQ_2 \cdot CQ_3} = \frac{BP_1 \cdot CP_2 \cdot AP_3}{BQ_1 \cdot CQ_2 \cdot AQ_3} = \frac{CP_1 \cdot AP_2 \cdot BP_3}{CQ_1 \cdot AQ_2 \cdot BQ_3}.$$

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* C. Adams's *Eigenschaften des...Dreiecks*, p. 8 (1846). Adams gives also (3).

§ 4.

Positions of two isogonal points with reference to a triangle.

(1) Any point on a side has for isogonal point the opposite vertex.

(2) A vertex has for isogonal point any point on the opposite side.

(3) A point inside the triangle has its isogonal point also inside the triangle.

(4) If a point be outside the triangle and situated in the angle vertically opposite to $\angle$ BAC, for example, its isogonal point will be outside the triangle and situated in that segment of the circumcircle (remote from A) cut off by BC.

(5) If a point be outside the circumcircle and situated within the angle BAC, for example, its isogonal point will be outside the circumcircle and situated within the same angle.

(6) If a point be on the circumference of the circumcircle, its isogonal point will be at infinity.

The truth of these statements,* which are not quite obvious, may be ascertained by the construction of a few figures. Of the last statement the following proof may be given :—

FIGURE 39.

If AD, BE, CF, be three parallel lines drawn through the vertices of a triangle ABC, their three isogonals will be concurrent at a point on the circumference of the circumcircle.†

Because AD, BE, CF are parallel,

therefore arc AE = arc BD, arc BC = arc EF.

Make arc CP equal to arc BD; join AP, BP, CP.

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* They are all given by Mr J. J. A. Mathieu in *Nouvelles Annales*, 2nd series, IV. 403 (1865).

† Professor Eugenio Beltrami in *Memorie del l'Accademia delle Scienze del Istituto di Bologna*, 2nd series, II., 383 (1863).

Since $\quad$ arc CP = arc BD,
therefore $\quad \angle$ CAP = $\angle$ BAD,
and AP is isogonal to AD.

Since $\quad$ arc CP = arc AE,
therefore $\quad \angle$ CBP = $\angle$ ABE,
and BP is isogonal to BE.

Since arc BC = arc EF, arc CP = arc AE,
therefore $\quad$ arc BP = arc AF;
therefore $\quad \angle$ BCP = $\angle$ ACF,
and CP is isogonal to CF.

Hence, if P be a point on the circumcircle of ABC, the point isogonal to it is the point of concurrency of AD, BE, CF.

(1) AD is perpendicular* to the Wallace line P (ABC).

This follows from § 1, (3).

§ 5.

If three angular transversals cut the opposite sides in three collinear points, their isogonals will also cut the opposite sides in three collinear points.†

FIGURE 40.

Let AD, AD′; BE, BE′; CF, CF′ be pairs of isogonals; then if D, E, F, be collinear, so will D′, E′, F′.

For
$$\frac{BD \cdot BD'}{CD \cdot CD'} = \frac{c^2}{b^2},$$
$$\frac{CE \cdot CE'}{AE \cdot AE'} = \frac{a^2}{c^2},$$
$$\frac{AF \cdot AF'}{BF \cdot BF'} = \frac{b^2}{a^2};$$
therefore
$$\frac{BD \cdot CE \cdot AF}{CD \cdot AE \cdot BF} \cdot \frac{BD' \cdot CE' \cdot AF'}{CD' \cdot AE' \cdot BF'} = 1.$$

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* Professor J. Neuberg in Rouché and Comberousse's *Traité de Géometrié*, First Part, p. 439 (1891).

† Townsend's *Modern Geometry*, I. 181 (1863).

Now $$\frac{BD \cdot CE \cdot AF}{CD \cdot AE \cdot BF} = 1;$$

therefore $$\frac{BD' \cdot CE' \cdot AF'}{CD' \cdot AE' \cdot BF'} = 1;$$

therefore D′, E′, F′ are collinear.

§ 6.

If O be any point in the plane of triangle ABC, and AO BO CO meet the circumcircle in A_1 B_1 C_1 *and D E F be the projections of O on BC CA AB the triangles* $A_1B_1C_1$ *DEF are directly similar, and the point O of triangle DEF corresponds to that point of* $A_1B_1C_1$ *which is isogonal* to O.*

FIGURE 41.

For the points O F B D are concyclic;

therefore $$\angle FDO = \angle FBO = \angle B_1A_1O.$$

Similarly $$\angle EDO = \angle C_1A_1O.$$

The demonstration may be easily seen to apply to the more general case where A_1 B_1 C_1 are taken inverse to O with any other constant of inversion.*

(1) If O be the orthocentre of ABC, it must be the incentre or an excentre of DEF, and therefore the incentre or an excentre of $A_1B_1C_1$.

(2) If O be the circumcentre of ABC, it must be the orthocentre of DEF, and therefore the circumcentre of $A_1B_1C_1$.

(3) If O be the incentre of ABC, it must be the circumcentre of DEF, and therefore the orthocentre of $A_1B_1C_1$.

(4) If O be an excentre of ABC, it must be the circumcentre of DEF, and therefore the orthocentre of $A_1B_1C_1$.

---

* Mr E. M. Langley and Professor Neuberg. (1)—(4) are Mr Langley's. See the *Seventeenth General Report of the Association for the Improvement of Geometrical Teaching*, p. 45 (1891.)

§ 7

*If two points be isogonal with respect to a triangle their six projections on the sides of the triangle are concyclic.**

FIGURE 42.

Let O, O′ be isogonal with respect to ABC, and let D, E, F, D′, E′, F′ be their projections on the sides BC, CA, AB.

Then EF is antiparallel to E′F′ with respect to A ;

therefore	E,E′ F,F′,	are concyclic.
Similarly	F,F′ D,D′	,, ,, ,
and	D,D′ E,E′	,, ,, ;

therefore the six points are concyclic.

§ 8

If O O′ be isogonal points with respect to ABC, and D E F D′E′F′ be their respective projections on BC CA AB, then

AO BO CO are perpendicular to the sides of D′E′F′
AO BO′ CO′ ,, ,, ,, D E F.

FIGURE 42.

This has been established in § 1, (3).

The application of the preceding properties of isogonals to the particular case of medians and symmedians will be taken up in a succeeding paper.

---

* Steiner in Gergonne's *Annales*, xix. 37-64 (1828).
In the same article will be found also the property of § 8.

Eighth Meeting, Friday, June 14th, 1895.

JOHN M'COWAN, Esq., M.A., D.Sc., President, in the Chair.

A Summary of the Theory of the Refraction of their approximately Axial Pencils through a Series of Media bounded by coaxial Spherical Surfaces, with Applications to a Photographic Triplet, &c.

By PROFESSOR CHRYSTAL.

[*The Paper will be published in the next Volume.*]

On a Diophantine Equation.

By R. F. DAVIS, M.A.

In the consideration of Question 12612 appearing in the *Educational Times* for January of this year, proposed by the Rev. Dr. Haughton, F.R.S., of Trinity College, Dublin, the following Diophantine Equation suggests itself:

What values of x make $8x^3 - 8x + 16 = \square$?

Since it may be written $8x(x^2-1)+16 = \square$ it is obvious that $x = 0_1 \pm 1$ are solutions. Also that $x = 2$ is a solution. Moreover $x = -\frac{3}{2}$ when substituted gives $-27+12+16=1$ and is therefore a solution,—marking approximately a limit to the negative root.

I. Put $8x^3 - 8x + 16 = (px^2 + x - 4)^2$; then after reduction and division by x^2, we have

$$p^1x^2 - 2x(4-p) + 1 - 8p = 0 \qquad \dots \qquad \dots \qquad \dots \qquad \text{(A)}$$

It will be found that the roots of this equation are real and rational when $\qquad 8p^3 - 8p + 16 - \square$

which is the same Diophantine Equation as that with which we started.

Hence the values of x obtained by experiment may be used for p in the equation (A) with the certainty of obtaining one or more fresh solutions.

Thus put $p = 0$ and we get	$-\ 8x + 1 = 0$	$x = \frac{1}{8}$	
,, ,, $p = 1$,, ,, ,,	$x^2 - 6x + 7 = 0$	$x = -1$ or 7	
,, ,, $p = 1$,, ,, ,,	$x^2 - 10x + 9 = 0$	$x = +1$ or 9	
,, ,, $p = 2$,, ,, ,,	$4x^2 - 4x + 15 = 0$	$x = \frac{5}{2}, \frac{3}{2},$	

all depending on the fact that if one root of a quadratic equation be real and rational, so is the other root.

II. The equation (A) may be written

$$(px + 1)^2 = 8(x + p)$$
$$= 16a^2, \quad \text{say};$$

whence $\qquad px + 1 = 4a, \quad$ and $\quad x + p = 2a^2.$

Thus
$$x(2a^2 - x) - 4a + 1 = 0$$
$$x^2 - 2a^2x + 4a - 1 = 0 \qquad \ldots \qquad \ldots \qquad (B)$$

and the roots of this equation are real and rational when

$$a^4 - 4a + 1 = \square.$$

Any value of x satisfying the original problem will, if substituted in (B), give two real and rational values of a. If one of these values of a be substituted in (B) and the equation then solved as regards x we get the original value of x and another value.

Thus we are led (somewhat blindly it is true) to an interminable series of solutions: such as

$$0, 1, -1, 2, 7, 9, 15, 496$$

$$\frac{5}{2}, \quad -\frac{3}{2}, \quad \frac{1}{8}, \quad \frac{26}{9}, \quad -\frac{7}{9}, \quad \frac{17}{25}, \quad -\frac{38}{25}, \quad \frac{39}{49}, \quad -\frac{55}{49}, \quad \frac{71}{81}, \quad \text{etc.}$$

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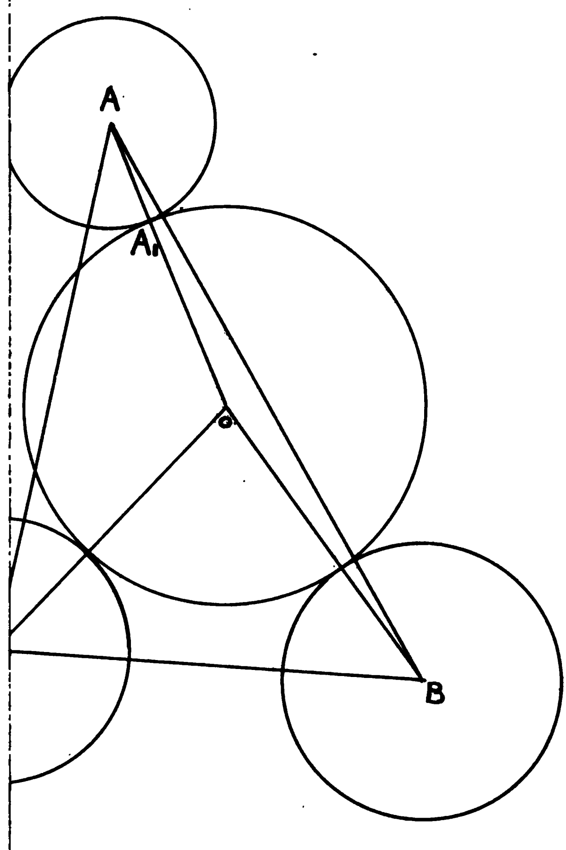
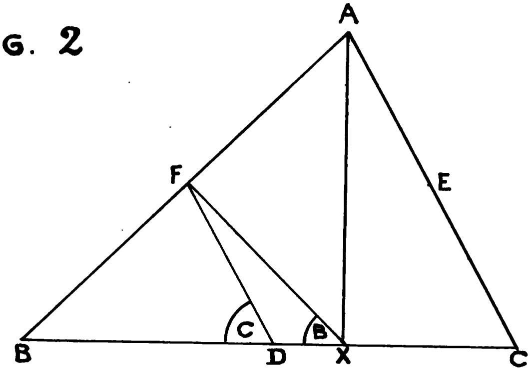


FIG. 2



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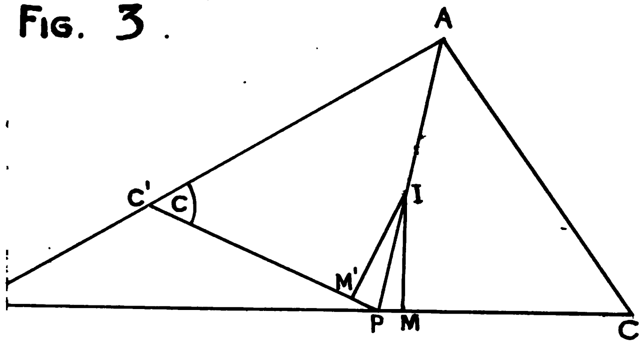

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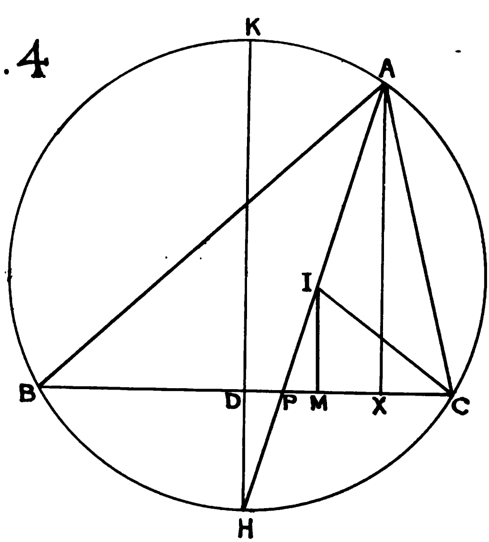

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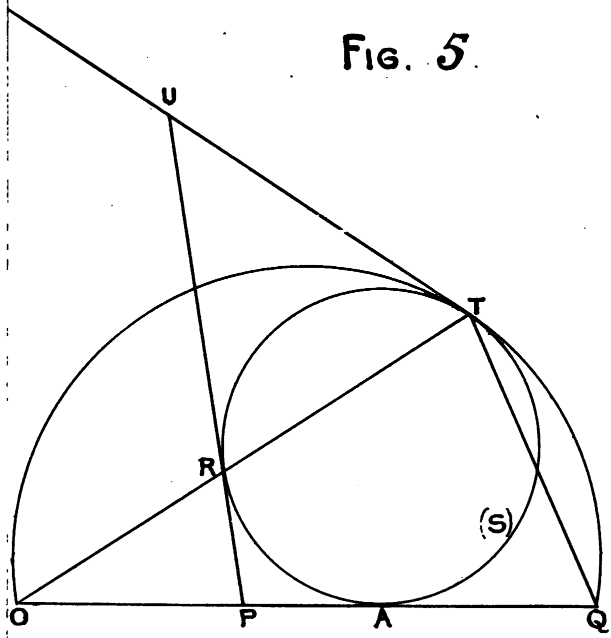

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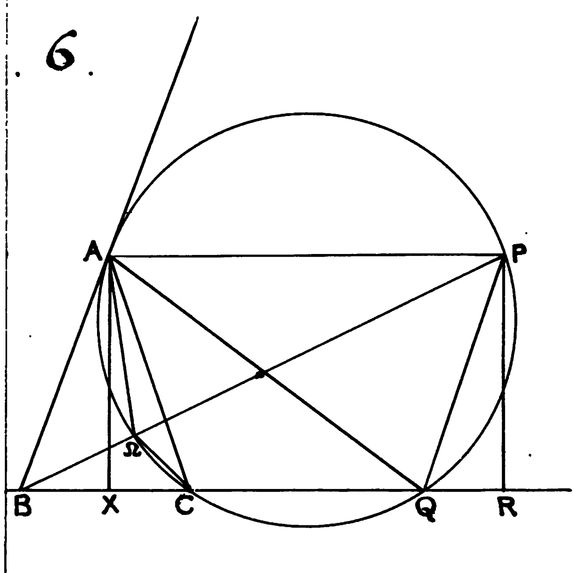

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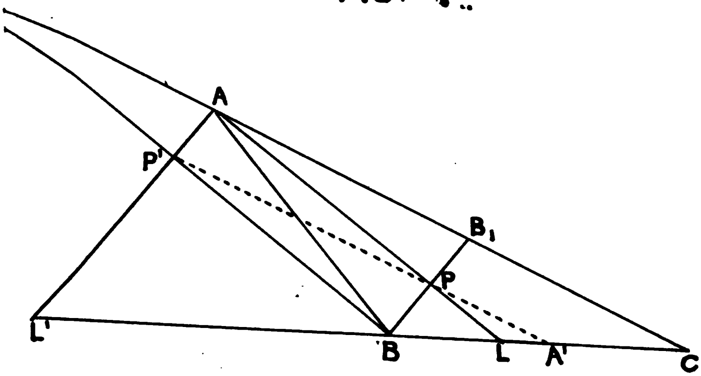

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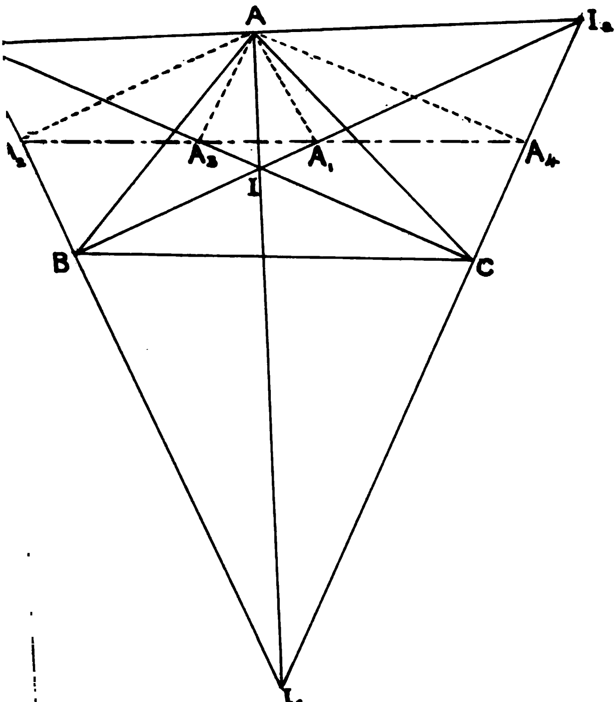


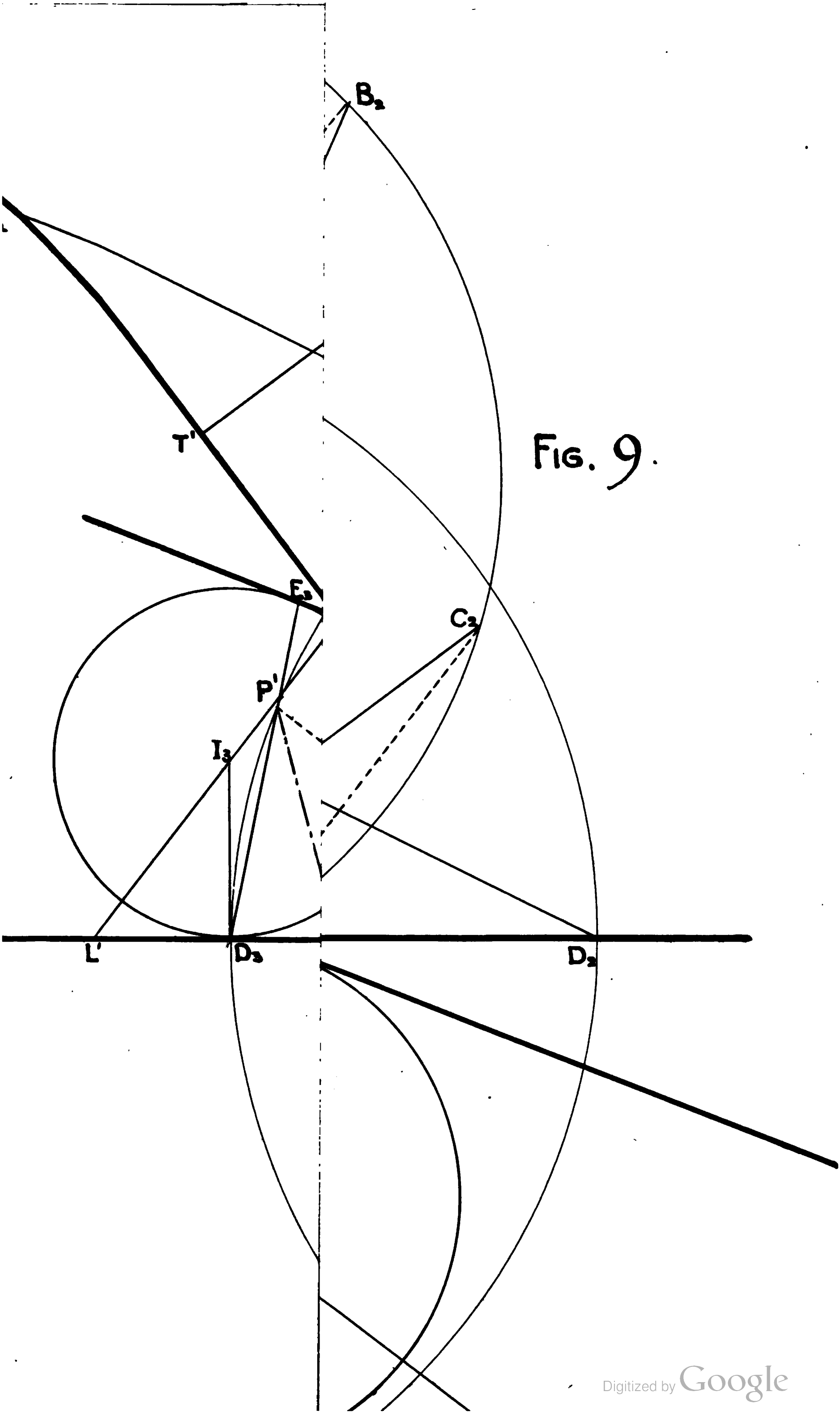


FIG. 9.

FIG. 10.

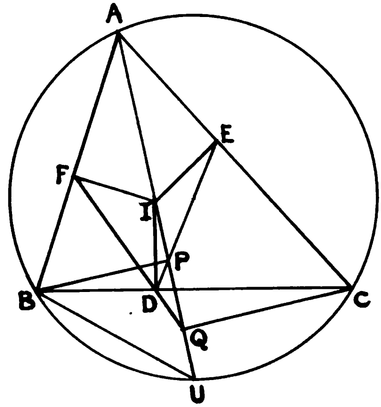


FIG. 11

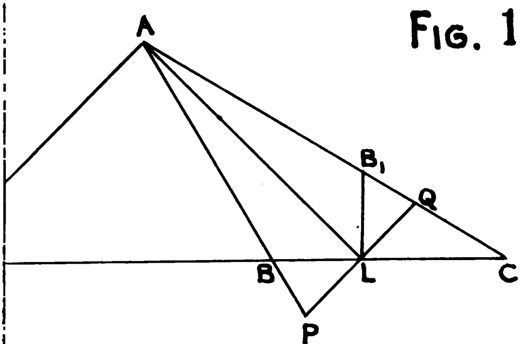
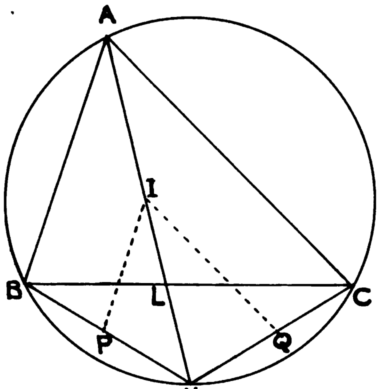


FIG. 12



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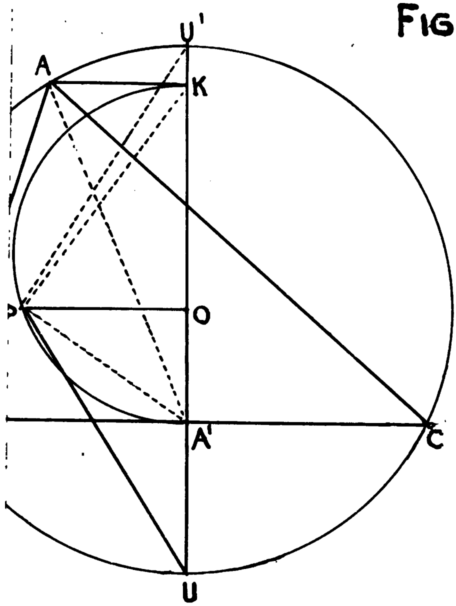
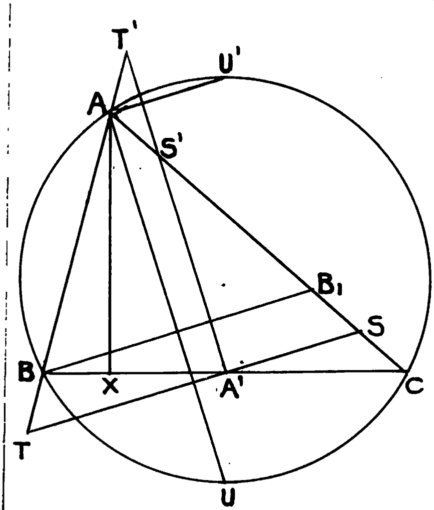


FIG 14



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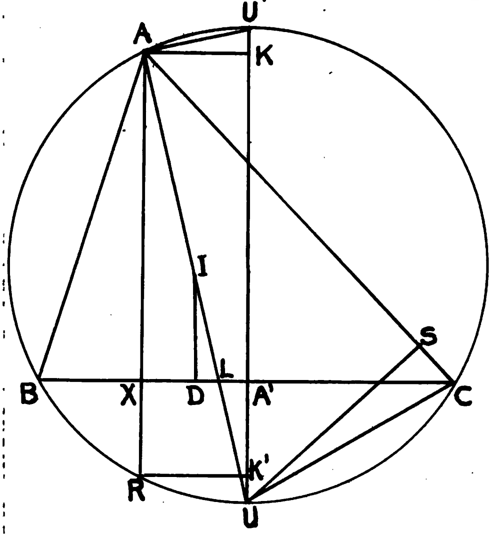
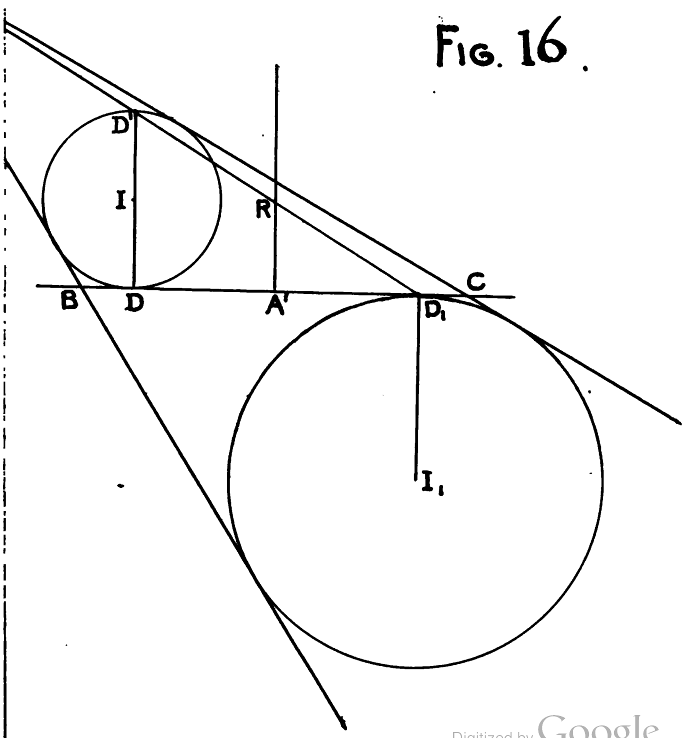


FIG. 16.



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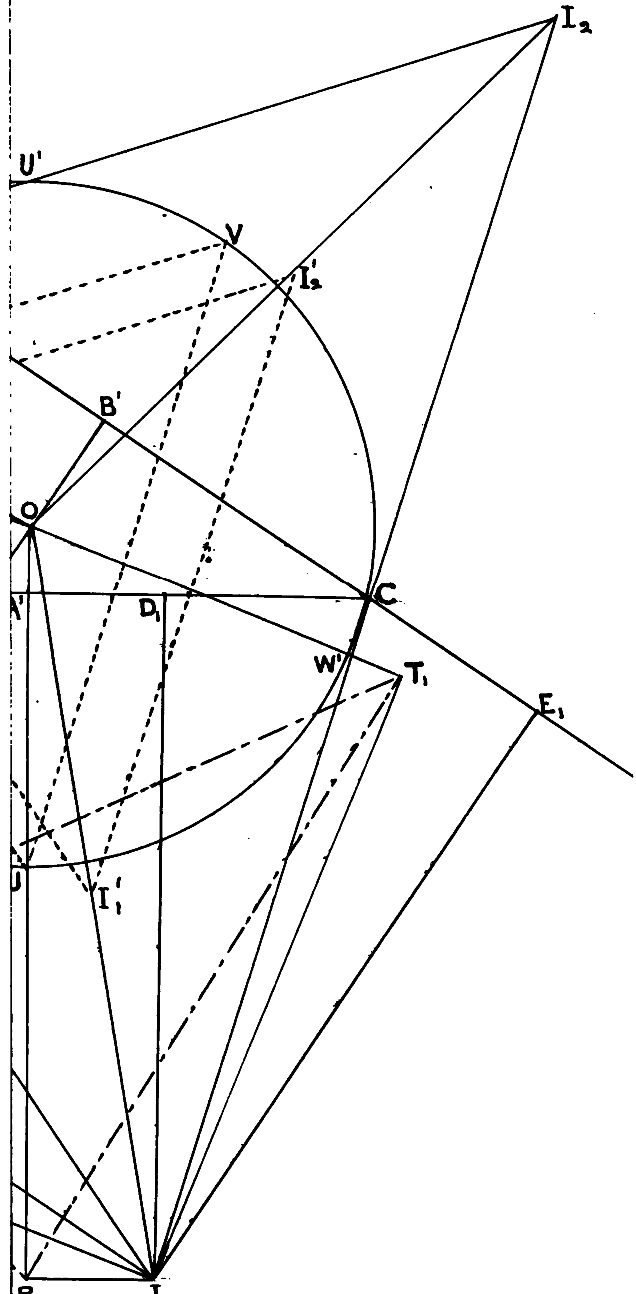


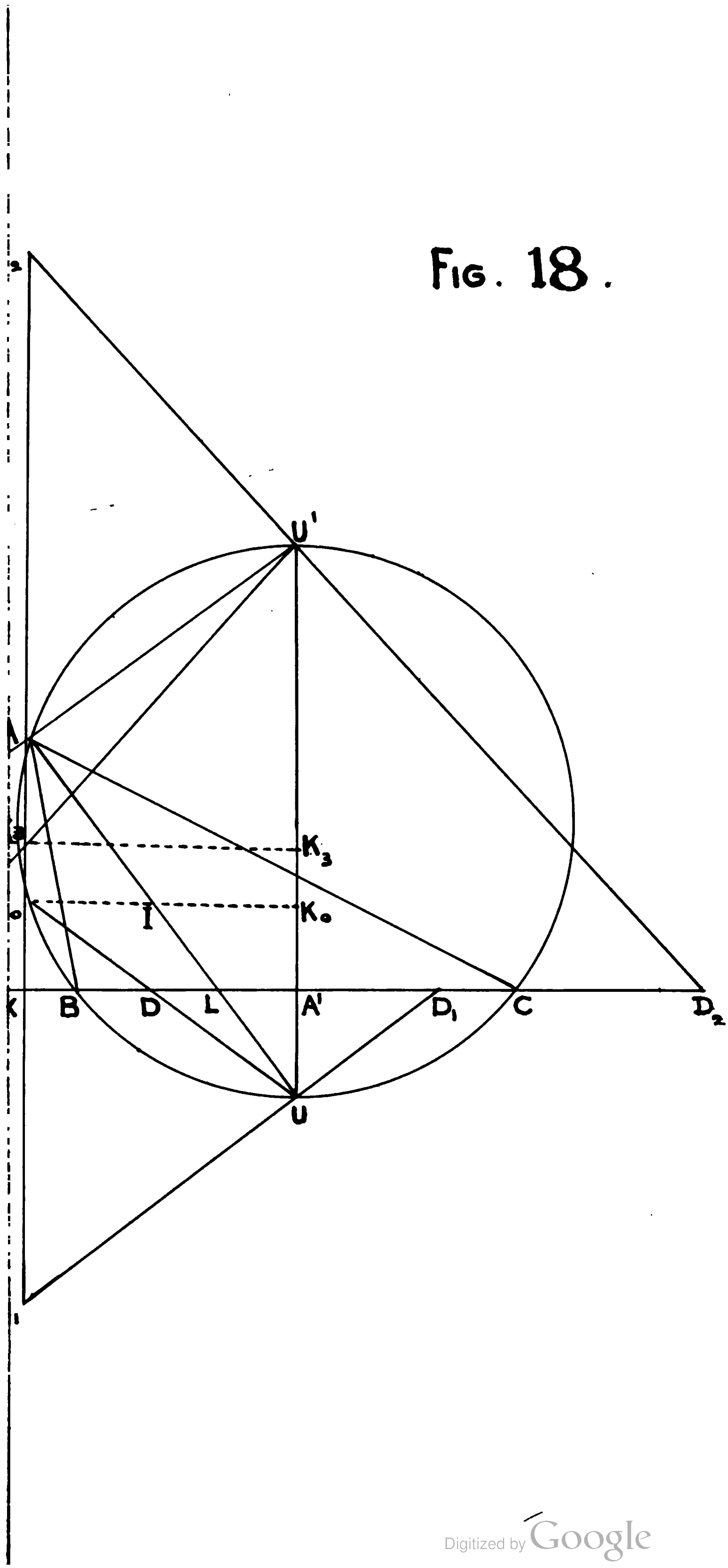
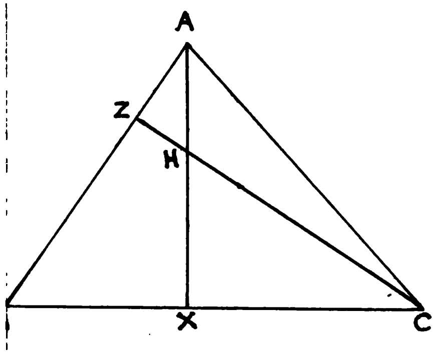


FIG. 18.

FIG. 20



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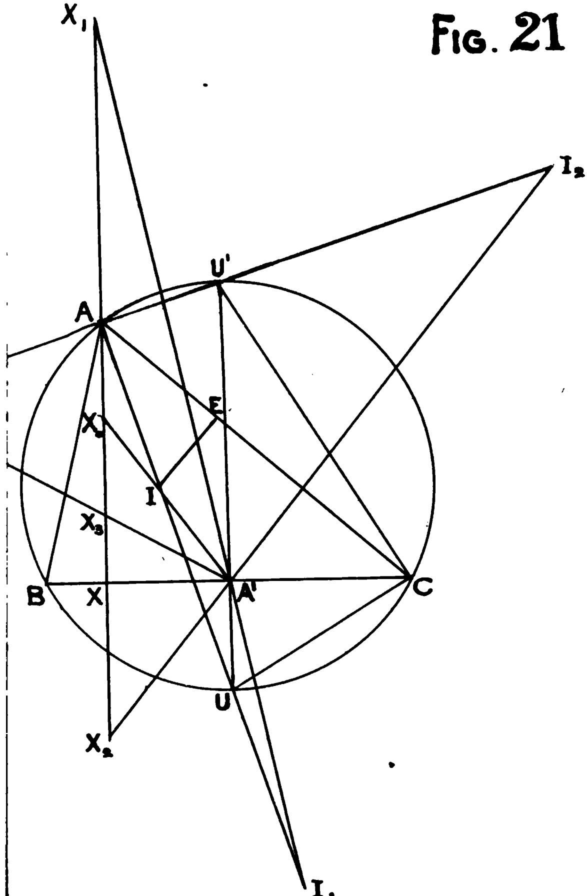

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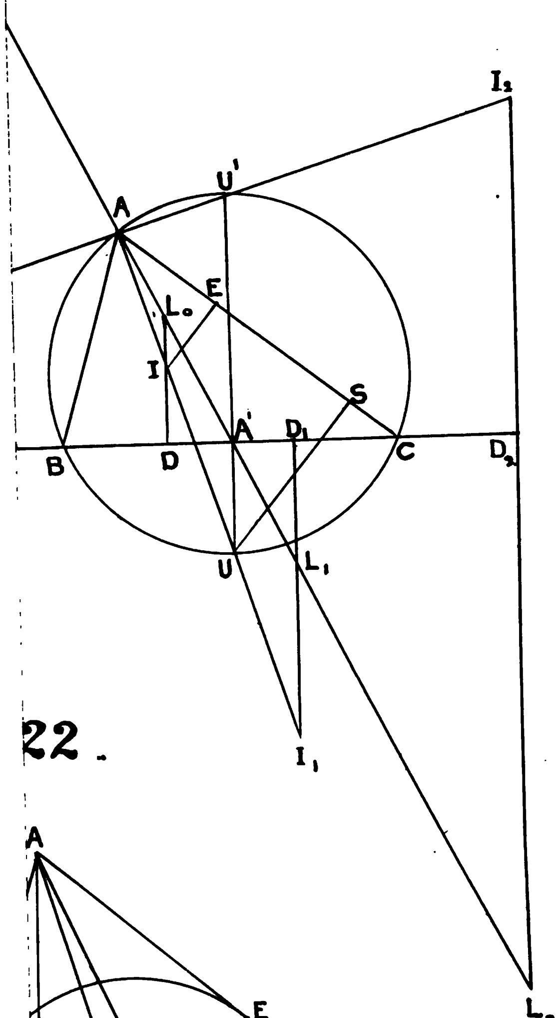

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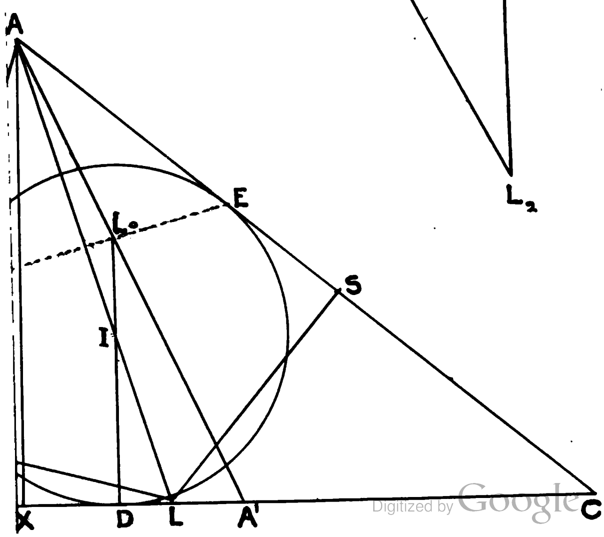

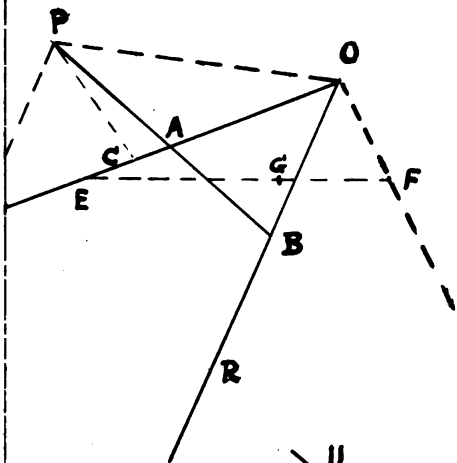
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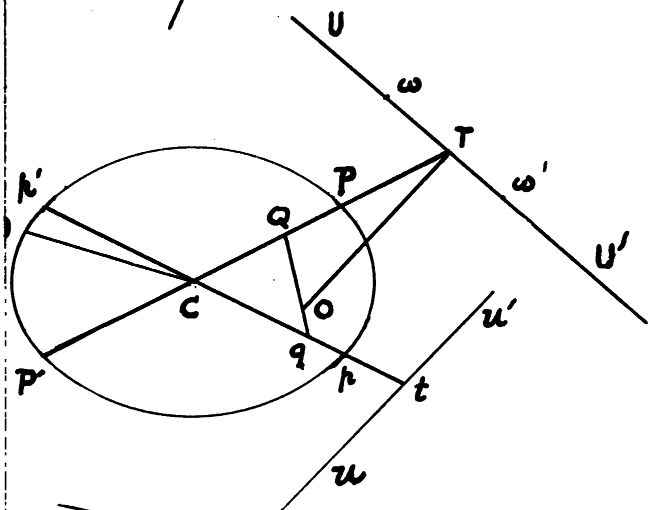
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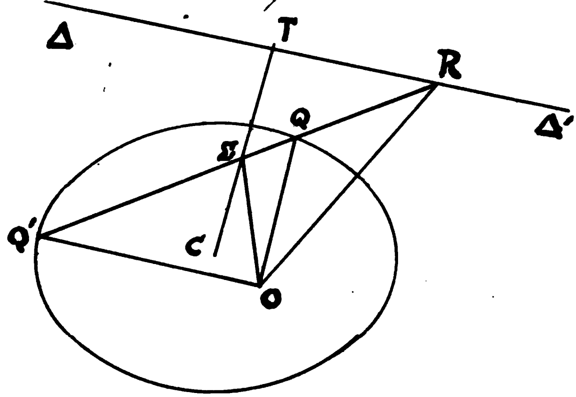
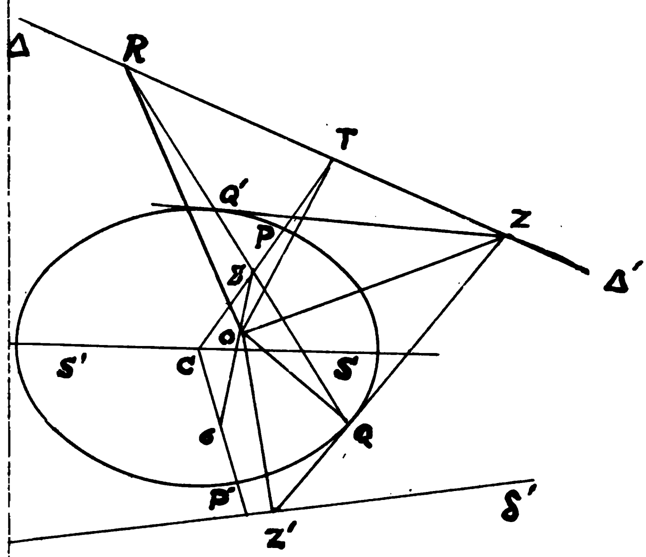
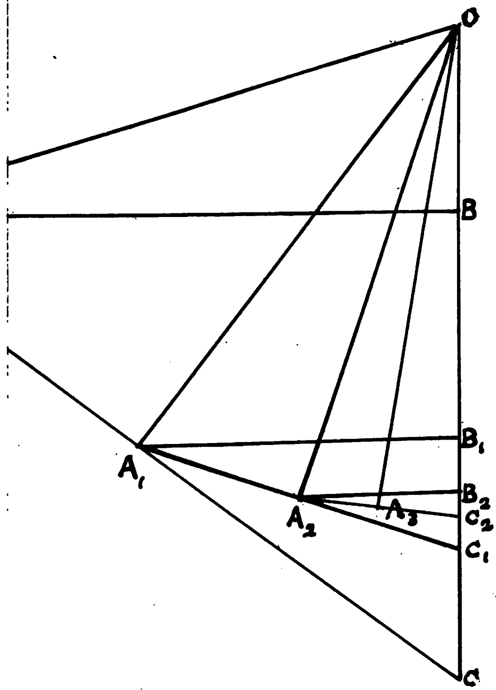
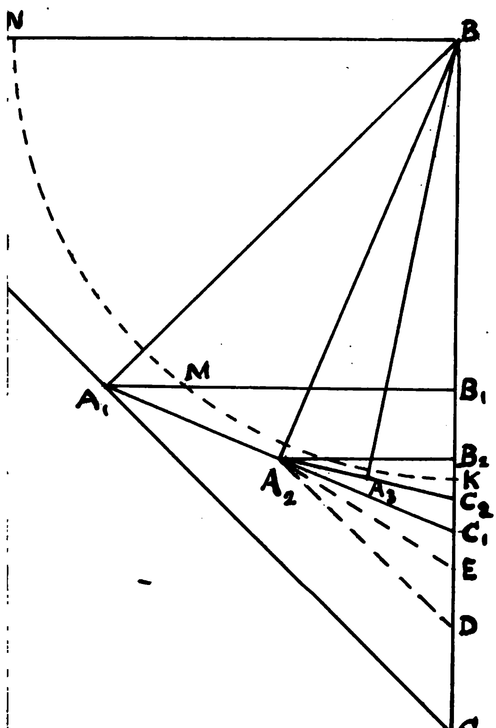

T
Δ
R
Q
Δ'
Σ
Q'
C
O



Δ
R
T
Q′
Z
Δ′
P
δ
O
S′
C
S
Q
σ
P′
δ′
Z′


O
B
B_1
A_1
B_2
A_3
A_2
C_2
C_1
C


N
B
M
A_1
B_1
B_2
K
A_2
A_3
C_2
C_1
E
D
C

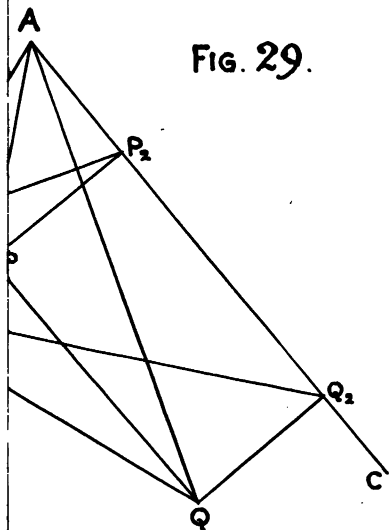


Fig. 29.

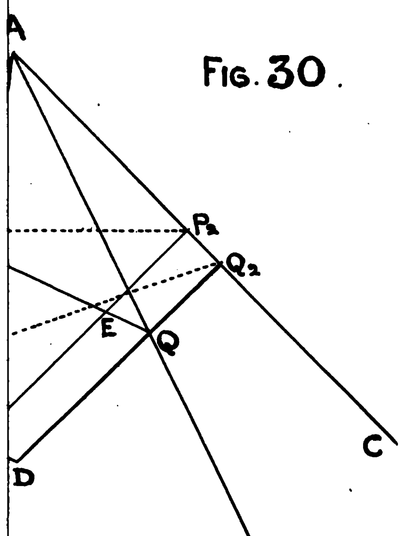


Fig. 30.

FIG. 29.

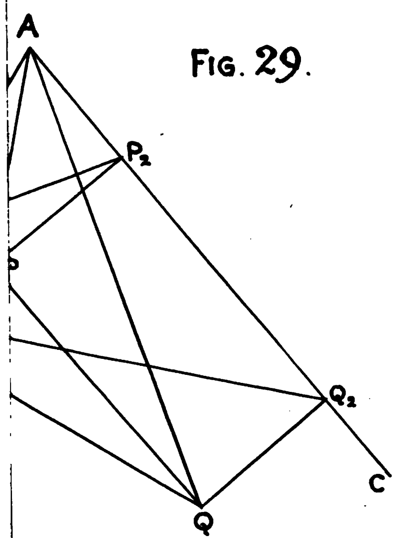
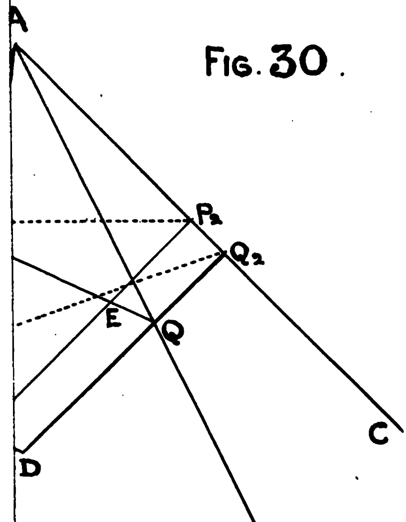


FIG. 30.



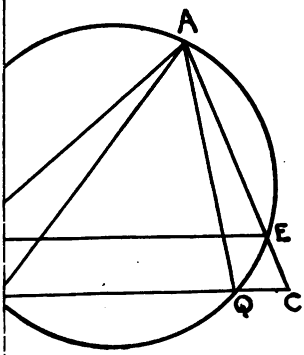


FIG. 31.

32.

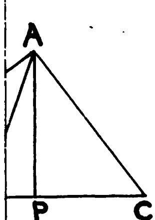
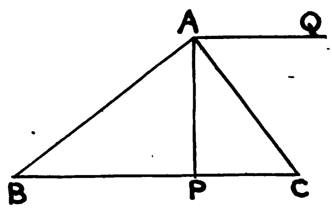


FIG. 33.



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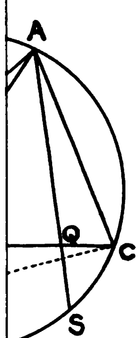


FIG. 35.

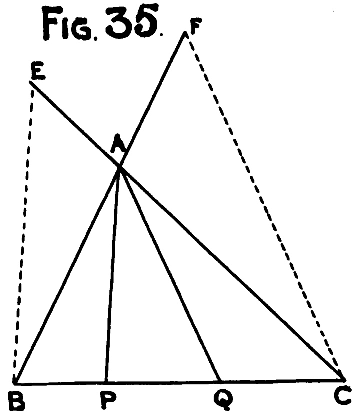
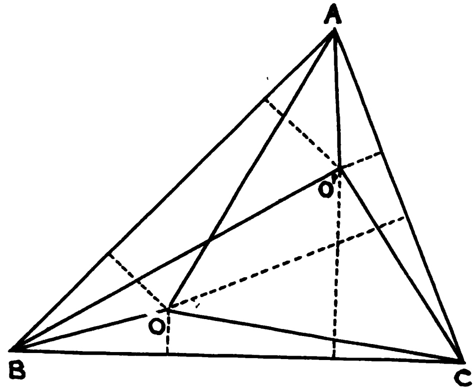


Fig. 37.



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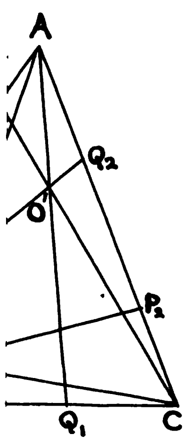
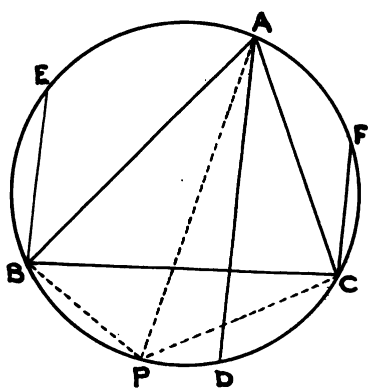


Fig. 39.



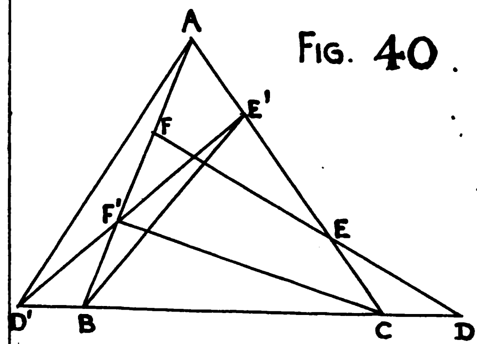


FIG. 40.

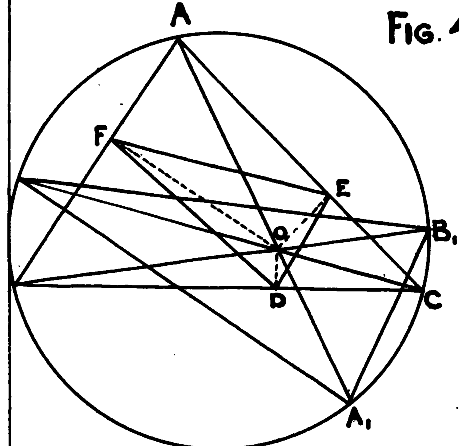


FIG. 41.

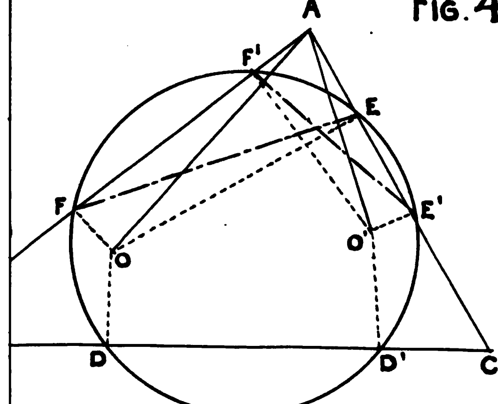


FIG. 42.